

MATH 248A. A LATTICE IN THE ADELE RING

Let  $K$  be a number field, and let  $S$  be a finite set of places of  $K$  containing the set  $S_\infty$  of archimedean places. The  $S$ -adele ring  $\mathbf{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$  is an open subring of the adèle ring  $\mathbf{A}_K$ , and it meets the additive subgroup  $K$  in the subring  $\mathcal{O}_{K,S}$  of  $S$ -integers. In class we saw that  $K$  is discrete in  $\mathbf{A}_K$  and  $\mathcal{O}_{K,S}$  is discrete in  $\mathbf{A}_{K,S}$ . The purpose of this handout is to explain why each of these discrete subgroups is co-compact. The main input will be that  $\mathcal{O}_{K,S}$  is a lattice in  $\prod_{v \in S} K_v$ , a fact that we saw in an earlier handout (and that was a key ingredient in the proof of the  $S$ -unit theorem).

1. THE CASE OF  $S$ -ADELES

We first study  $\mathbf{A}_{K,S}/\mathcal{O}_{K,S}$ . This is a locally compact Hausdorff topological group since  $\mathbf{A}_{K,S}$  is such a group and  $\mathcal{O}_{K,S}$  is a closed subgroup. Using the continuous projection to the  $S$ -factor, we get a continuous surjection

$$\mathbf{A}_{K,S}/\mathcal{O}_{K,S} \twoheadrightarrow \left( \prod_{v \in S} K_v \right) / \mathcal{O}_{K,S}$$

to a compact group, and the kernel is surjected onto by the compact group  $\prod_{v \notin S} \mathcal{O}_v$ , so the kernel is itself compact. Hence, to infer compactness of  $\mathbf{A}_{K,S}/\mathcal{O}_{K,S}$  we use the rather general:

**Lemma 1.0.1.** *Let  $G$  be a locally compact Hausdorff topological group, and let  $H$  be a closed subgroup. Give the coset space  $G/H$  its locally compact Hausdorff quotient topology. The group  $G$  is compact if and only if  $H$  and  $G/H$  are compact.*

*Proof.* If  $G$  is compact then its closed subset  $H$  is compact and its continuous image  $G/H$  is certainly compact. For the converse, choose a collection of open sets  $U_i$  in  $G$  with compact closure in  $G$  (such  $U_i$ 's exist since  $G$  is locally compact and Hausdorff). The images  $U_i H/H$  in  $G/H$  are open since  $U_i H$  is open in  $G$  (as it is a union of translates  $U_i h$  for  $h \in H$ ), and so by compactness of  $G/H$  finitely many of these cover  $G/H$ . That is,  $G$  is a union of finitely many  $U_i H$ 's. Each  $U_i$  has compact closure  $C_i$  in  $G$ , so  $G$  is a union of finitely many  $C_i H$ 's. Each  $C_i H$  is compact because it is the image of the compact product  $C_i \times H$  under the continuous multiplication map for  $G$ , so  $G$  is a union of finitely many compact subsets. Hence,  $G$  is compact. ■

2. THE ADELIC CASE

Now we turn to the task of proving that  $\mathbf{A}_K/K$  is compact. For any  $S$ ,  $\mathbf{A}_{K,S}$  is an open subring of  $\mathbf{A}_K$ , so the continuous map

$$\mathbf{A}_{K,S}/\mathcal{O}_{K,S} \rightarrow \mathbf{A}_K/K$$

modulo discrete subgroups is an open map. It is also injective (why?), and so it is an open embedding. Hence,  $\mathbf{A}_K/K$  is the directed union of open subgroups  $\mathbf{A}_{K,S}/\mathcal{O}_{K,S}$  that are compact. It follows that  $\mathbf{A}_K/K$  is compact if and only if these open embeddings are equalities for  $S$  "large enough".

Let us try to understand the cokernel of the inclusion, namely  $\mathbf{A}_K/(K + \mathbf{A}_{K,S})$ . This is a discrete topological group (since  $\mathbf{A}_{K,S}$  is an open additive subgroup of  $\mathbf{A}_K$ ), and from the definitions we have

$$\mathbf{A}_K/\mathbf{A}_{K,S} = \bigoplus_{v \notin S} K_v/\mathcal{O}_v.$$

We have a natural diagonal embedding of  $K$  in here with kernel given by the set of elements of  $K$  that are  $v$ -adically integral for all  $v \notin S$ , which is to say the kernel of the diagonal map from  $K$  is  $\mathcal{O}_{K,S}$ . Hence, we have an inclusion

$$K/\mathcal{O}_{K,S} \hookrightarrow \bigoplus_{v \notin S} K_v/\mathcal{O}_v,$$

and our aim is to prove that this is an equality for  $S$  "large enough". Since  $\mathcal{O}_{K,S}$  can always be arranged to have trivial class group by taking  $S$  to be large enough (for example,  $S$  consists of  $S_\infty$  and non-archimedean places corresponding to prime factors of a finite set of representatives for the elements of the class group of  $\mathcal{O}_K$ ), it suffices to prove that this inclusion is an equality whenever  $\mathcal{O}_{K,S}$  has trivial class group. Rather more generally:

**Theorem 2.0.1.** *Let  $A$  be an arbitrary Dedekind domain, with  $F$  its fraction field. The natural inclusion*

$$F/A \rightarrow \bigoplus_{\mathfrak{m}} F_{\mathfrak{m}}/A_{\mathfrak{m}}^{\wedge}$$

*is an equality if  $A$  has trivial class group. Here,  $\mathfrak{m}$  ranges over the maximal ideals of  $A$ ,  $F_{\mathfrak{m}}$  is the  $\mathfrak{m}$ -adic completion of  $F$ , and  $A_{\mathfrak{m}}^{\wedge}$  is its valuation ring.*

Taking  $A = \mathcal{O}_{K,S}$  gives what we need, since the completions from the primes of  $\mathcal{O}_{K,S}$  are exactly the  $K_v$ 's and  $\mathcal{O}_v$ 's for  $v \notin S$  (check!). Also, the map into the direct sum in the theorem makes sense because each  $x \in F$  has image in  $F_{\mathfrak{m}}$  that lies in  $A_{\mathfrak{m}}^{\wedge}$  for all but finitely many  $\mathfrak{m}$ .

Before reading the proof of the theorem, it is instructive to first study the case  $A = \mathbf{Z}$  by hand to get a feel for what is going on: can you show that  $\mathbf{Q}/\mathbf{Z} \rightarrow \bigoplus_p \mathbf{Q}_p/\mathbf{Z}_p$  is an isomorphism “by hand”? (hint:  $\mathbf{Q}/\mathbf{Z}$  is a torsion group, so it maps isomorphically to the direct sum of its  $p^{\infty}$ -torsion subgroups.)

*Proof.* We first get rid of completions by checking that the natural map

$$F/A_{\mathfrak{m}} \rightarrow F_{\mathfrak{m}}/A_{\mathfrak{m}}^{\wedge}$$

is an isomorphism, where  $A_{\mathfrak{m}}$  denotes the algebraic localization of  $A$  at  $\mathfrak{m}$ ; this is the special case of the theorem when  $A$  is a discrete valuation ring. That is, for any discrete valuation ring  $R$  with fraction field  $L$  we claim that the natural map

$$L/R \rightarrow L^{\wedge}/R^{\wedge}$$

is an isomorphism, with  $L^{\wedge}$  denoting the completion of  $L$  with respect to the canonical place arising from  $R$  and  $R^{\wedge}$  denoting its valuation ring. The injectivity of this map is the old result  $L \cap R^{\wedge} = R$  inside of  $L^{\wedge}$ , a fact we saw long ago. For surjectivity, we pick  $\pi \in R$  that is a uniformizer and we choose a subset  $\Sigma \subseteq R$  that is a set of representatives for the residue field of  $R$ , which in turn coincides with the residue field of the discrete valuation ring  $R^{\wedge}$ . By using  $\pi$ -adic expansions with coefficients in  $\Sigma$ , consideration of  $L^{\wedge}/R^{\wedge}$  is tantamount to looking at just the “polar parts” of  $\pi$ -adic expansions in  $L^{\wedge}$ . Each such polar part is a *finite* sum  $\sum_{i>0} \sigma_i \pi^{-i} \in L$ , so the asserted surjectivity from  $L/R$  follows.

We may now restate our problem in purely algebraic terms: we wish to show that the natural inclusion

$$F/A \rightarrow \bigoplus_{\mathfrak{m}} F/A_{\mathfrak{m}}$$

is an isomorphism if  $A$  has trivial class group. Assume  $A$  has trivial class group. The target is generated by elements that are 0 away from a single factor, so it suffices to fix  $\mathfrak{m}$  and a nonzero  $\mathfrak{m}$ -adic polar part  $\xi \in F/A_{\mathfrak{m}}$ , and we have to construct  $x \in F$  that is integral away from  $\mathfrak{m}$  but has  $\mathfrak{m}$ -adic polar part  $\xi$ . If  $\xi$  has  $\mathfrak{m}$ -adic order  $-e$  with  $e > 0$  and  $\pi \in A$  is a principal generator of  $\mathfrak{m}$ , then there is a well-defined residue class

$$\pi^e \xi \in A_{\mathfrak{m}}/\mathfrak{m}^e A_{\mathfrak{m}} \simeq A/\mathfrak{m}^e$$

(the isomorphism goes from right to left, and is the crux of the matter). For any  $x' \in A$  that represents this residue class in  $A/\mathfrak{m}^e$ ,  $x'/\pi^e \in F$  solves our problem: it is integral away from  $\mathfrak{m}$  and has polar part  $\xi$  at  $\mathfrak{m}$ . ■

We conclude with a concrete example to illustrate the compactness of  $\mathbf{A}_K/K$  in the special case  $K = \mathbf{Q}$ . We claim

$$\mathbf{A}_{\mathbf{Q}} = \mathbf{Q} + [0, 1] \times \prod_p \mathbf{Z}_p,$$

which is to say that the compact subset  $[0, 1] \times \prod_p \mathbf{Z}_p \subseteq \mathbf{A}_{\mathbf{Q}}$  (with the induced topology given by its product topology!) maps continuously onto the quotient  $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ , thereby “explaining” the compactness of the quotient in this case. For an arbitrary adele  $x = (x_v) \in \mathbf{A}_{\mathbf{Q}}$ , for each prime  $p$  we let  $y_p = \sum_{i>0} a_i/p^i \in \mathbf{Q}$  (with  $a_i \in \mathbf{Z}$ ,  $0 \leq a_i < p$ ) be the finite-tailed  $p$ -adic polar part of  $x_p \in \mathbf{Q}_p$ , so  $y_p \in \mathbf{Z}[1/p]$  is integral away from  $p$  for each  $p$  and  $y_p = 0$  for all but finitely many  $p$  (as  $x_p \in \mathbf{Z}_p$  for all but finitely many  $p$ ). Hence, if  $\ell \neq p$  are distinct primes then  $y_{\ell}$  has image in  $\mathbf{Q}_p$  that lies in  $\mathbf{Z}_p$ , so the rational number  $y = \sum_p y_p \in \mathbf{Q}$  makes sense and its image in  $\mathbf{Q}_p/\mathbf{Z}_p$  is represented by  $y_p$  for each  $p$ . Thus,  $x_p - y \in \mathbf{Q}_p$  lies in  $\mathbf{Z}_p$  for every  $p$

because modulo  $\mathbf{Z}_p$  it is represented by  $x_p - y_p \in \mathbf{Z}_p$  (due to the choice of  $y_p$ ). To summarize, the diagonally embedded adèle  $y$  in  $\mathbf{A}_{\mathbf{Q}}$  differs from the adèle  $x$  by an adèle that is integral at *all* non-archimedean places. That is,

$$x - y = (x_v - y) \in \mathbf{R} \times \prod_p \mathbf{Z}_p \subseteq \mathbf{A}_{\mathbf{Q}}.$$

If we further replace  $y$  by a  $\mathbf{Z}$ -translate then we do not affect this property of  $x - y$ , but we can adjust the archimedean component of the difference to lie in  $[0, 1]$ . That is, for a suitable integer  $n$  we have  $x - (y + n) \in [0, 1] \times \prod_p \mathbf{Z}_p$ , giving the desired result. This argument shows slightly more:  $[0, 1] \times \prod_p \mathbf{Z}_p$  maps bijectively onto  $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ , so it serves as a “fundamental domain”.

*Remark 2.0.2.* The preceding worked example generalizes to  $\mathbf{A}_K$  for any number field  $K$ , saying that if  $P \subseteq \mathbf{R} \otimes_{\mathbf{Q}} K = \prod_{v|\infty} K_v$  is a fundamental parallelotope for the quotient  $(\mathbf{R} \otimes_{\mathbf{Q}} K)/\mathcal{O}_K$  then  $\mathbf{A}_K = K + P \times \prod_{v \nmid \infty} \mathcal{O}_v$ , so  $P \times \prod_{v \nmid \infty} \mathcal{O}_v$  serves as a “fundamental domain” for  $\mathbf{A}_K/K$  (showing in another way that this Hausdorff quotient is compact, being the continuous image of a compact subset of  $\mathbf{A}_K$ ).

Indeed, if  $\alpha \in \mathbf{A}_K$  then there is just a finite set  $T$  of finite places  $v$  for which  $\alpha_v \notin \mathcal{O}_v$ , say with  $\text{ord}_v(\alpha_v) = -n_v < 0$  for each  $v \in T$ . If  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_K$  corresponding to a chosen  $v \in T$  then  $\mathfrak{m}^{-n_v}$  is dense in  $\mathfrak{m}_v^{-n_v} \subset K_v$ , so we can pick  $a_v \in \mathfrak{m}^{-n_v}$  with  $|\alpha_v - a_v| \leq 1$ . Note that such  $a_v \in K$  is integral at all finite places away from  $v$ . Hence, the element  $x = \sum_{v \in T} a_v \in K$  satisfies  $|\alpha_v - x|_v \leq 1$  for all  $v \in T$  as well as for all finite  $v \notin T$  (since  $\alpha_v, x \in \mathcal{O}_v$  for all finite  $v \notin T$ ), so  $\alpha - x \in \prod_{v|\infty} K_v \times \prod_{v \nmid \infty} \mathcal{O}_v$ . Since  $P$  is a fundamental domain for  $\mathcal{O}_K$  in  $\mathbf{R} \otimes_{\mathbf{Q}} K = \prod_{v|\infty} K_v$ , we can find  $c \in \mathcal{O}_K$  such that the archimedean factor  $(\alpha - x)_{\infty}$  differs from  $c$  by an element of  $P$ . Thus,  $\alpha - (x + c) \in P \times \prod_{v \nmid \infty} \mathcal{O}_v$ , with  $x + c \in K$ .

In practice, working with adelic constructions is the best way to bypass undue complications caused by class groups.