# Math 216A. Homework 9

*"... the usual definition of a scheme is not nicely suited to our proof."* Nagata

Ch. II: 4.4 (omit the noetherian hypothesis!), 4.6*, 4.11, 5.1, 5.6, 5.7, 5.8.

For 4.6, more generally a proper affine map between noetherian schemes is finite: reduce to the case of integral affine schemes and use [H, Ch. II, Thm. 4.11A] = [Mat, Thm. 10.4]. For 4.11(a), see [Mat, Thm. 11.7] (and corollary) for a proof of the Krull-Akizuki Theorem.

For 5.1, also construct natural isomorphisms of $\mathscr{O}_X$-modules

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{G}) \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G})), \;\; (\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F})^\vee \simeq \mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}^\vee.$$

**Exercise A**. For a morphism $f : X \to S$, an $S$-scheme $S'$, and the base change $f' : X' \to S'$, prove: if $f$ is proper then so is $f'$, and the converse holds if $S' \to S$ is fpqc. (The converse is *false* for "projective" since it is not local on the base.) Hint: don't use the valuative criterion.

**Exercise B**. For $r \geq 1$, a *rank-$r$ vector bundle* on a locally ringed space $(X, \mathscr{O}_X)$ is an $\mathscr{O}_X$-module $\mathscr{E}$ that is "locally free of rank $r$": for some open cover $\{V_j\}$ of $X$, $\mathscr{E}|_{V_j} \simeq \mathscr{O}_{V_j}^{\oplus r}$ as $\mathscr{O}_{V_j}$-modules for all $j$. When $r = 1$ we call $\mathscr{E}$ a *line bundle* or an *invertible* sheaf. (For schemes $X$, such $\mathscr{E}$ are clearly quasi-coherent.)

(i) If $\mathscr{F}$ is an $\mathscr{O}_X$-module and $q : \mathscr{F} \twoheadrightarrow \mathscr{E}$ and $q' : \mathscr{F} \twoheadrightarrow \mathscr{E}'$ are rank-$r$ vector bundle quotients, show there is at most one $\mathscr{O}_X$-linear map $f : \mathscr{E}' \to \mathscr{E}$ satisfying $f' \circ q' = q$ and that such an $f$ (if it exists) is an isomorphism (we then say $(\mathscr{E}, q)$ and $(\mathscr{E}', q')$ are *isomorphic*). Deduce that there is a *set* of isomorphism classes of rank-$n$ vector bundle quotients of $\mathscr{F}$.

(ii) For $n \geq 0$, explain why an invertible quotient of $\mathscr{O}_X^{\oplus(n+1)}$ is the "same thing" as data $(\mathscr{L}, (s_0, \dots, s_n))$ consisting of an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and an ordered $n$-tuple of global sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ such that for all $x \in X$ some $s_j(x) \in \mathscr{L}(x) := \mathscr{L}_x/\mathfrak{m}_x\mathscr{L}_x$ is nonzero. How is an isomorphism of invertible quotients expressed in terms of such data?

(iii) For a ring $R$, integer $n \geq 0$, and $0 \leq i \leq n$, let the functors $P_{n+1}, U_{n,i} : \mathrm{Sch}_R \rightrightarrows \mathrm{Set}$ be defined as follows: $P_{n+1}(X)$ is the set of isomorphism classes as in (ii) and $U_{n,i}(X) \subset P_{n+1}(X)$ is the subset of such data for which $s_i(x) \neq 0$ for all $x \in X$; these are contravariant functors via *pullback* of sheaves. For any $(\mathscr{L}, (s_0, \dots, s_n)) \in U_{n,i}(X)$ show $\mathscr{O}_X \to \mathscr{L}$ via $f \mapsto fs$ is an isomorphism, so for $j \neq i$ we have $s_j = f_j s_i$ for a unique $f_j \in \mathscr{O}_X(X)$. Use this to prove $U_{n,i}$ is represented by $D_+(T_i) = \mathbf{A}_R^n$.

(iv) For $\xi = (\mathscr{L}, (s_0, \dots, s_n)) \in P_{n+1}(X)$, show $X_i = \{x \in X \mid s_i(x) \neq 0\}$ is the maximal open $V \subset X$ for which $\xi|_V \in U_{n,i}(V)$ and that the $X_i$'s cover $X$. Deduce $P_{n+1}$ is represented by $\mathbf{P}_R^n$, and thereby define an *injection* $\{(a_0, \dots, a_n) \in A^{n+1} \mid a_i\text{'s generate } (1)\}/A^\times \hookrightarrow \mathbf{P}^n(A)$ for $R$-algebras $A$, surjective if all line bundles on $\mathrm{Spec}(A)$ are free (e.g., $A$ local!).

(v) By (iv), over $\mathbf{P}_R^n$ there is a "universal structure", denoted $(\mathscr{O}(1), (T_0, \dots, T_n))$. Describe it in terms of gluing over the $n+1$ standard open affine $n$-spaces.

(vi) Define the *Segre map* $S_{n,m} : \mathbf{P}_R^n \times_R \mathbf{P}_R^m \to \mathbf{P}_R^{(n+1)(m+1)-1}$ on $X$-valued points by

$$((\mathscr{L}, (s_0, \dots, s_n)), (\mathscr{L}', (s'_0, \dots, s'_m))) \mapsto (\mathscr{L} \otimes \mathscr{L}', (s_i \otimes s'_j))$$

(fix an enumeration of the $(n+1)(m+1)$ ordered pairs of indices $(i, j)$). Why does the right side make sense in $P_{(n+1)(m+1)}(X)$? Prove $S_{n,m}$ is a closed immersion by studying the preimage of each $D_+(T_{(i,j)})$ (show it is $D_+(T_i) \times_R D_+(T_j)$). Describe $S_{n,m}$ on $A$-valued points for *local* $R$-algebras $A$ via the end of (iv) to reprove $S_{n,m}$ is proper via the valuative criterion.