Math 216A. Homework 8

"You can wave your hands as much as you want, but it will never make you fly." M. Kisin

Exercise A. A ring map $\varphi : A \to B$ is faithfully flat if and only if it is flat and $\mathrm{Spec}(\varphi)$ is surjective (see [Mat, Thm. 7.3] with $M = B$), so a map of schemes $X \to S$ is called *faithfully flat* if it is flat and surjective.

(i) If $X \to S$ is flat (resp. faithfully flat), show so is $X \times_S S' \to S'$ for any S-scheme S'.

(ii) If $S' \to S$ is faithfully flat and quasi-compact (fpqc: *fidèlement plat et quasi-compact*), prove S acquires the quotient topology from S'. (Hint: Use that a finite disjoint union of affines is affine to reduce to the affine case, and check the quotient topology using closed sets rather than open sets. If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is faithfully flat with reduced A and a dense $\Sigma \subset \mathrm{Spec}(A)$ has closed preimage $\mathrm{Spec}(B/I)$ with radical I, show $A \to B/I$ is injective and study the image of $\mathrm{pr}_1 : \mathrm{Spec}(B \otimes_A (B/I)) = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B/I) \to \mathrm{Spec}(B)$.)

(iii) If $A \to B$ is flat and X is a quasi-compact quasi-separated A-scheme, for $X_B := X \times_A B$ show the natural map $\Gamma(X, \mathscr{O}_X) \otimes_A B \to \Gamma(X_B, \mathscr{O}_{X_B})$ is an isomorphism. (Hint: Express global sections as a certain kernel involving *finite* direct products.)

(iv) If $A \to A'$ is faithfully flat and B is an A-algebra such that $A' \otimes_A B$ is finitely generated over A' prove B is finitely generated over A. (Hint: Write A'-algebra generators as finite sums of elementary tensors.)

(v) (fpqc descent for properties of morphisms) Let $f : X \to S$ be a morphism, and let $f' : X' \to S'$ be its base change by an fpqc map. Prove f has property $\mathbf{P}$ if f' does, where $\mathbf{P}$ is: quasi-compact, quasi-separated, open immersion, locally of finite type (use (iv)), finite type, quasi-finite, universally injective, surjective, flat, affine (use quasi-separatedness and (iii)), closed immersion, separated, finite, integral (use affineness and apply "finite" with finite type subalgebras), isomorphism. (Hint: reduce to S and S' both affine.)

Remark. A deeper part of descent theory is to give criteria on an fpqc map $S' \to S$ and S'-scheme X' that specify an S-scheme X (the "descent") and an S'-isomorphism $\theta : X \times_S S' \simeq X'$ (and similarly for "descending" S'-morphisms to S-morphisms). It is beyond the level of this course, but the technique involves $S' \times_S S'$ (as a "generalized overlap for gluing") and this is often *not* noetherian even when S and S' are noetherian but $S' \to S$ is not of finite type (or a localization thereof). A typical such case is $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(\widehat{A})$ for a noetherian local ring A and completion $\widehat{A}$, since $\widehat{A} \otimes_A \widehat{A}$ is rarely noetherian. This is one of the reasons for trying to avoid noetherian hypotheses in the foundations.

Exercise B. Let C be a category with a final object $*$ and direct products for all pairs of objects. (In the category of S-schemes, S is a final object and fiber products over S are direct products in this category.) A *group in* C is a 4-tuple (G, m, e, i) consisting of an object G and morphisms $m : G \times G \to G$ ("multiplication"), $e : * \to G$ ("identity"), and $i : G \to G$ ("inversion") such that (with $\pi : G \to *$ the unique map) the following diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(e\circ\pi, \mathrm{id}_G)} & G \times G \\ {\scriptstyle (\mathrm{id}_G, e\circ\pi)}\downarrow & \searrow\!\!= & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(i, \mathrm{id}_G)} & G \times G \\ {\scriptstyle (\mathrm{id}_G, i)}\downarrow & \searrow^{e\circ\pi} & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array}$$

In the category of sets, $*$ is the 1-point set and so these axioms recover the usual concept.

(i) For an object G, show that specifying a group structure on it in C is the same as giving the set $G(T) := \mathrm{Hom}(T, G)$ a group structure functorially in T.

(ii) A *homomorphism* is a map $f : G \to H$ compatible with m's, e's, and i's in the evident diagram sense; by Yoneda's Lemma, it is equivalent that for all T the map $G(T) \to H(T)$ is a group homomorphism. Show e and i are uniquely determined by m, and that a map $f : G \to H$ compatible with m_G and m_H (i.e., $m_H \circ (f \times f) = f \circ m_G$) must respect e's and i's. (In private, to appreciate Yoneda's Lemma, try to do this exercise without it!)

(iii) For a ring A, consider the functor $T \rightsquigarrow \mathrm{GL}_n(\Gamma(T, \mathscr{O}_T))$ on A-schemes. Prove this is represented by $\mathrm{GL}_{n,A} = \mathrm{Spec}(A[x_{ij}]_{\det})$ (with $1 \le i, j \le n$) and write down the A-algebra maps corresponding to m, e, i (justified via Yoneda's Lemma).

(iv) Assuming C admits fiber products, the *kernel* of $f : G \to H$ is $\ker f := G \times_{H,e} *$. Show its functor of points is $T \rightsquigarrow \ker(G(T) \to H(T))$, and in the category of A-group schemes for any ring A compute the coordinate rings of $\mathrm{SL}_{n,A} := \ker(\det : \mathrm{GL}_{n,A} \to \mathrm{GL}_{1,A})$ and $\mu_{m,A} := \ker(t^m : \mathrm{GL}_{1,A} \to \mathrm{GL}_{1,A})$ for $m \ge 1$ (defined functorially in the evident manner).

Remark. For any finite flat map $X \to Y$ to a locally noetherian scheme, the function $r(y) = \dim_{k(y)} \mathscr{O}(X_y)$ is locally constant on Y (why?); it is called the *rank* of X over Y. For a finite set Σ, the functor $\mathrm{Sch}_Y \to \mathrm{Set}$ of locally constant Σ-valued functions is represented by $\underline{\Sigma}_Y := \coprod_{\sigma \in \Sigma} Y$ which has $r(y) = \#\Sigma$ for all $y \in Y$. If $\pi : G \to Y$ is a finite flat group scheme with constant fiber-rank N (called its *order*, since for $G = \underline{\Gamma}_Y$ for a finite group Γ, this is $\#\Gamma$), by analogy with Cauchy's theorem on finite groups being killed by their order it is natural to wonder if the Y-morphism $G \to G$ defined by $g \mapsto g^N$ is trivial (i.e., equal to $e_G \circ \pi$). For commutative G this is true and is due to Deligne (proved on the bus going to his year of service in the Belgian army); in the non-commutative case it is unsolved except when Y is the spectrum of a field. The commutative case is extremely useful in number theory.

Exercise C. Let X_1, X_2 be S-schemes, U_i open in X_i, and $j : U_1 \simeq U_2$ an S-isomorphism, so the gluing X of X_1 and X_2 along j is an S-scheme.

(i) Prove X is separated over S if and only if X_1 and X_2 are S-separated and the graph morphism $\Gamma_j : U_1 \to X_1 \times_S X_2$ is a closed immersion. (Hint: study $\Delta_{X/S} : X \to X \times_S X$ using the open cover $\{X_i \times_S X_{i'}\}_{1 \le i, i' \le 2}$ of $X \times_S X$.)

(ii) Let $S = \mathrm{Spec}(k)$ for a field k, $X_1 = X_2 = \mathbf{A}^1_k$ and $U_1 = U_2 = \mathbf{A}^1_k - \{0\}$. Let $j : U_1 \simeq U_2$ be the identity map and $j' : U_1 \simeq U_2$ be $t \mapsto 1/t$, and denote the respective gluings as X and X', so X is the "line with doubled origin" and $X' = \mathbf{P}^1_k$. Use the graph criterion in (i) to explain why X is not separated and X' is separated.

Exercise D. (Correction of [H, Ch. II, Exer. 3.11(d)]) Let $f : X \to Y$ be quasi-compact and quasi-separated. Show the closure $\overline{f(X)}$ admits a closed subscheme structure $Z \hookrightarrow Y$ such that (i) it is initial among closed subschemes of Y through which f factors, and (ii) for open $U \subset Y$, the scheme-theoretic image of $f^{-1}(U) \to U$ is $Z \cap U$. Also check that the associated ideal sheaf $\mathscr{I}_Z \subset \mathscr{O}_Y$ is $\ker(\mathscr{O}_Y \to f_*(\mathscr{O}_X))$. [One calls Z the *scheme-theoretic image* of f, and for f a quasi-compact immersion it is called the *schematic closure* of X in Y.]

Hint: if $\{U_i\}$ is an open cover of Y for which $f_i : f^{-1}(U_i) \to U_i$ admits such a $Z_i \subset U_i$ satisfying the analogues of (i) and (ii) for f_i, show the Z_i's glue to a solution to (i) and (ii) for f; Exercise A(iii) and the Nike trick are useful for handling affine Y. For the equality of *subsheaves* at the end, it suffices to equate sets of sections over a base of opens (why?).