Math 216A. Homework 6

"...The study of category theory for its own sake (surely one of the most sterile of all intellectual pursuits) also dates from this time. Grothendieck himself can't necessarily be blamed for this, since his own use of categories was very successful in solving problems."

Miles Reid, in *Undergraduate Algebraic Geometry.*

Read the Proj handout, noting in particular Examples 3.4 and 3.5, and Section 4.

Ch. II: 3.4 (hint: use the Nike trick to reduce to the case $Y = \mathrm{Spec}(B)$, and use Exercise 2.17(b) from HW5), 3.5 (require f to also be of finite type; this is essential for the notion of quasi-finiteness to possess certain good properties, as we will see later), 3.6* (assume $U \neq \emptyset$, and note that if X is just irreducible and locally noetherian, then $\mathscr{O}_\xi$ is a local artin ring), 3.7* (if X is also irreducible, observe that f is dominant if and only if f takes the generic point of X to the generic point of Y), 3.8 (assume $U \neq \emptyset$, and see [Mat, p. 64, Remark]), 3.11(b), 3.17 (omit (f)), 3.18, 3.19, 3.20* (omit (f) and assume only that X is locally of finite type over k, and note that in (a), necessarily $\dim X < \infty$).

Exercise A. Say a map of schemes $f : X \to Y$ is *affine* if Y is covered by affine open subschemes $\mathrm{Spec}(B_i)$ such that each open preimage $f^{-1}(\mathrm{Spec}\, B_i)$ is affine. Show that if f is affine then for *every* affine open $\mathrm{Spec}\, B \subset Y$, the preimage open subscheme $f^{-1}(\mathrm{Spec}\, B) \subset X$ is affine. (Hint: use the Nike trick and Exercise 2.17(b) from HW5.)

Exercise B. Let R be any ring, $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ for R-algebras A and B with $A \simeq R[T_1, \dots, T_n]/I$ for a finitely generated ideal I (one says A is "of finite presentation" over R; when R is noetherian this is of course equivalent to saying A is finitely generated as an R-algebra).

(i) For $x \in X$ and $y \in Y$, to show that every local R-algebra map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ arises from an R-scheme map $f : U \to Y$ carrying x to y for an open $U \subset X$ around x. This is called "spreading out" for morphisms. (Hint: seek basic affine open U via denominator-chasing and mapping properties of localization and polynomial rings.)

(ii) If $f' : U' \to Y$ is a second such map then show $f|_{U''} = f'|_{U''}$ for some open $U'' \subset U \cap U'$ around x.

(iii) If X and Y are also integral, with respective function fields $K(X)$ and $K(Y)$, by taking x and y to be the respective generic points show that R-algebra maps $K(Y) \to K(X)$ correspond to equivalence classes of dominant R-scheme maps $f : U \to Y$ for non-empty open $U \subset X$ (where f is *equivalent* to $g : V \to Y$ when f and g coincide on a non-empty open subset of $U \cap V \neq \emptyset$; this is an equivalence relation – especially transitive – because all non-empty open subsets of an irreducible space meet each other due to density of each). Such $f : U \to Y$ are often called "dominant rational maps".