Math 216A. Holiday Homework 10 (purely for fun)

It is very difficult nowadays to write books on mathematics. If one does not take care of the real subtleties of propositions, explanations, proofs, and conclusions, the book will not be mathematics: but if one does this, reading becomes a bore. Kepler (1609).

Ch. II: 3.8 (U non-empty there), 5.10 (and (e): for saturated I, show $\mathrm{Proj}(S/I)$ is reduced if and only if I is radical, and is integral if and only if I is prime), 5.15(a,b), 5.16, 5.17

For 5.16(c), the natural map goes in the direction $S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'') \to F^p/F^{p+1}$, and you may find it helpful to first prove that there is a canonical isomorphism

$$\bigoplus_{k=0}^{n} (S^k(M) \otimes_A S^{n-k}(N)) \simeq S^n(M \oplus N)$$

for finite free A-modules M and N.

For 5.17, it is easier to prove more generally that for $f : X \to Y$ and quasi-coherent $\mathscr{O}_Y$-algebras $\mathscr{A}$, there is a bifunctorial identification $\mathrm{Hom}_Y(X, \mathbf{Spec}(\mathscr{A})) \simeq \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{A}, f_*(\mathscr{O}_X))$, generalizing [H, Ch. II, Exer. 2.4] (in other words, affine maps are "the same" as quasi-coherent algebras on the base). First construct a *natural map* between the Hom-sets, then work locally to see it is bijective (observing that both sides give sheaves of sets on Y in an appropriate manner).

Exercse A. Carry out the following more precise version of Exercise 5.18.

A *vector scheme over* Y is a commutative Y-group scheme X equipped with a morphism $\mathbf{A}^1_Y \times_Y X \to X$ making the group $X(T) = \mathrm{Hom}_Y(T, X)$ a module over $\Gamma(T, \mathscr{O}_T) = \mathbf{A}^1_Y(T)$ for all Y-schemes T (can write as commutative diagrams via Yoneda).

(i) Explain in what way $\mathbf{A}^n_Y$ is a vector scheme over Y.

(ii) Define a *rank-n geometric vector bundle over* Y to be a vector scheme X over Y such that there is an open covering $\{Y_i\}$ of Y and for each i an isomorphism of Y_i-vector schemes $X|Y_i \simeq \mathbf{A}^n_{Y_i}$. If X and X' are vector schemes over Y, explain how to define a vector scheme structure on $X \times_Y X'$, called the *product* vector scheme (and if X and X' are geometric vector bundles with ranks n and n', show that the product is one with rank $n + n'$).

(iii) If X is a rank-n geometric vector bundle over Y, show that the sheaf of sets $U \rightsquigarrow \mathscr{E}_X(U) := X(U)$ on U naturally has the structure of a locally free $\mathscr{O}_Y$-module of rank n. Conversely, if $\mathscr{E}$ is a locally free sheaf of rank n on Y, give $\mathbf{Spec}(S(\mathscr{E}^\vee))$ the structure of a rank n vector bundle over Y (hint: think of mapping properties [Yoneda] in order to justify working locally on Y). Check these constructions are suitably inverse to each other, are compatible with base change, and in particular take (finite) direct sums of locally free sheaves (of finite rank) over to products of geometric vector bundles.

Exercise B. (This is essentially Exercise 5.12.) For a scheme Y and separated finite type map $f : X \to Y$, an invertible sheaf $\mathscr{L}$ on X is called *very ample* over Y if $\mathscr{L} \simeq j^*(\mathscr{O}(1))$ for a Y-immersion $j : X \hookrightarrow \mathbf{P}^n_Y$. (The definition in [EGA II, 4.4.2] is more general, and even local on the base, but by [EGA II, 4.4.7] it agrees with this definition when Y is affine.)

(i) If invertible $\mathscr{L}$ and $\mathscr{N}$ on X are very ample over Y then show $\mathscr{L} \otimes \mathscr{N}$ is too.

(ii) If $g : Y \to Z$ is separated and finite type with invertible $\mathscr{L}$ on X very ample over Y and invertible $\mathscr{N}$ on Y very ample over Z, show $\mathscr{L} \otimes f^*\mathscr{N}$ on X is very ample over Z.