# Math 216A. Homework 1

*"As long as Algebra and Geometry were separated, their progress was slow and their use limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together towards perfection."*

Lagrange (1795)

In this and later homeworks, exercises labeled with an asterisk here are *not to be turned in* (they are included to provide useful practice, and you should always *at least read the statement* to be aware of what it is asking). The asterisk label in the textbook is not relevant to this.

Ch. I: 1.1 (char $k = 2$ is allowed, and in (c) it is meant that $W = \underline{Z}(f)$), 1.2, 1.3, 1.5* (statement should say "no nonzero nilpotent elements"), 1.6 (establish the second assertion, as the first was done in class), 1.7, 1.10, 1.11* (the *height* of a prime ideal $\mathfrak{p}$ in a commutative ring $A$ is the supremum of integers $n \geq 0$ for which there is a strictly increasing chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$, so it is $\dim A_{\mathfrak{p}}$ by another name), 1.12* (here the notation $\mathbf{A}^2_{\mathbf{R}}$ means $\mathbf{R}^2$ equipped with the "Zariski topology" defined by zeros of ideals in $\mathbf{R}[x, y]$, which will *not* be the meaning of such notation later on in the context of schemes, and to make the example interesting one should make it so that the zero locus of $f$ in $\mathbf{R}^2$ is non-empty).

For 1.11, a hint: thinking geometrically, note that the origin ought to be a singularity on the "curve" (once we give an algebraic definition of smoothness, this will be correct and will be the only such point). Look very carefully at this point, and view the ring as a $k[x]$-algebra.

**Some reading** (nothing to submit). If you have not already studied the important concept of "direct limit" for directed systems of modules and rings, teach yourself about it by looking in the following references (that the construction in the first two also works in the category of sets, making every set the direct limit of its finite subsets due to the universal mapping property, and in general "direct limit" is somewhat like an "abstract union" except that map to the direct limit from the the constituents of the directed system is generally *not* injective):

- Exercise 8 in Section 7.6 and Exercise 25 in Section 10.3 of Dummit & Foote for a concrete approach (with limited discussion of applications),
- the material on direct limits in Appendix A of Matsumura (especially the important Theorems A1 and A2 about the relation with tensor products and exactness),
- Exercises 14–22 in Chapter 2 of the famous but terse textbook "Introduction to Commutative Algebra" by Atiyah & MacDonald.