In §5 of Chapter 2 read up through Proposition 5.11 (exercises related to that will be given in HW10) and work on the exercises below.

**Remark.** As for the structure sheaf, a more elegant treatment of the construction $M \rightsquigarrow \tilde{M}$ assigning a sheaf of modules on $\text{Spec}(A)$ to an $A$-module $M$ is given in Qing Liu’s book: see §5.1.1–5.1.2 of Chapter 5 (and §5.1.4 for the analogue with graded modules on Proj). Also, beware that Hartshorne’s definition of coherent sheaf is wrong! It gives the right notion on (locally) noetherian schemes, but the correct viewpoint in general is the definition in §5.1.3 of Qing Liu’s book (which follows the definition of coherence first introduced by Serre in his important “FAC” paper). For schemes, coherence is generally not of interest beyond the locally noetherian case (though some important non-noetherian examples do come up in non-archimedean geometry). Complex analysis provides other significant examples of coherent sheaves in the correctly-defined sense (due to work of Oka).

**Extra 0.** *(i)* Let $X$ be a scheme locally of finite type over a field $k$. Assume $X \times_k K$ is irreducible (respectively reduced) for all separable (respectively purely inseparable) finite extensions $K/k$ (within a fixed algebraic closure of $k$, if you like). Prove that $X$ is geometrically irreducible (respectively reduced) over $k$ (hint: prove the contrapositive by trying to “descend the field” down from $k_s$ (resp. $k_p$)).

*(ii)* Let $f : X \to Y$ be a morphism between finite type schemes over a field. Assume that the fibers over all closed points are geometrically irreducible of dimension $d$. Prove that the fibers over all points of $Y$ are geometrically irreducible with dimension $d$ (hint: first reduce to the case in which the base field is algebraically closed, $X$ and $Y$ integral, $f$ is dominant, and fiber is the generic fiber).

There are many theorems along these lines which one needs in order to make completely rigorous the dictionary between abstract algebraic sets and schemes over an algebraically closed field; for example, geometric reducedness, etc. The essential point is to first show (local) constructibility of the set of points on a base scheme $S$ over which geometric fibers of a ‘reasonable’ morphism $f : X \to S$ have certain properties. This is (roughly) done by using a sort of ‘incidence correspondence’ and using Extra 2 below, analogously to the way one can show openness or closedness results using incidence correspondences when things are proper.

If $S$ is locally of finite type over a field $k$, then a constructible set $C$ in $S$ is completely determined by its closed points, and most reasonable properties of ‘geometric’ fibers are insensitive to the choice of algebraically closed base field, so over a closed point $s \in S$ we can just take the fiber over $k(s)$ to detect membership in $C$. Thus, for $k$ algebraically closed, in order to study $C$ it is often enough to consider the classical map $X(k) \to S(k)$.

**Extra 1.** *(i)* Let $X$ be a scheme. We say that $X$ is quasi-affine if there exists a quasi-compact (!) open immersion of $X$ into an affine scheme (i.e., $X$ is isomorphic to a quasi-compact (!) open in an affine). Show that $X$ is quasi-affine if and only if the canonical map $i_X : X \to \text{Spec}(\Gamma(X, \mathcal{O}_X))$ is a quasi-compact open embedding of topological spaces, in which case this canonical map is also an open immersion. Conclude in particular that $i_X$ is an isomorphism of schemes if and only if $i_X$ is a homeomorphism of topological spaces.
(The relationship between this definition – which is the one given initially in EGA – to more concrete versions when \( X \) is of finite type over an affine scheme will be taken up in a homework early in 216B.)

**Extra 2.** *(i)* We say that a subset \( Z \) in a topological space \( X \) is **constructible** if it is a finite union of sets of the form \( U_i \cap C_i \), where \( C_i = X \setminus V_i \) and \( U_i, V_i \) are open subsets of \( X \) for which the maps \( U_i, V_i \hookrightarrow X \) are quasi-compact. If instead each \( x \in X \) has an open neighborhood \( U \) so that \( Z \cap U \) is constructible in \( U \), then we say that \( Z \) is **locally constructible**. Prove that constructible and locally constructible sets are stable under formation of finite unions, finite intersections, and complements. Also, check that an open immersion \( U \hookrightarrow \text{Spec}(A) \) is quasi-compact if and only if the complement of \( U \) is of the form \( \text{Spec}(A/I) \) for a finitely generated ideal \( I \) in \( A \), and that in a quasi-compact, quasi-separated scheme, constructible sets and locally constructible sets are the same thing.

Let \( f : X \to S \) be a quasi-compact scheme morphism locally of finite presentation. Prove that the image of a locally constructible set is locally constructible (hint: reduce to the affine case and the study of \( f(X) \), and then reduce to the noetherian case, which you’ve settled). *(ii)* Prove that a locally constructible set in a scheme is open if and only if it is stable under specialization (hint: show that a constructible set in \( \text{Spec}(A) \) is the image of a map \( \text{Spec}(B) \to \text{Spec}(A) \); note this also applies to the complements of constructible sets).

*(iii)* Let \( f : X \to S \) be flat and locally of finite presentation. Prove that \( f \) is universally open, which is to say that for any base change \( S' \to S, f' \) is open. This is extremely important. For example, prove Extra 2 of HW8 for ‘flat’ in place of ‘proper’.

*(iv)* Let \( X \) be a \( k \)-scheme, \( k \) a field. Show that for any \( k \)-scheme \( Y, X \times_k Y \to Y \) is open. In particular, for any field extension \( K/k, K \times_k Y \to Y \) is an open map (hint: after reducing to the case \( X = \text{Spec}(A) \), and \( Y = \text{Spec}(B) \), note that any basic open set \( U \in X \times_k Y \) is the preimage of a basic open \( U' \) in \( X \times_k Y' \) for \( Y' = \text{Spec}(B') \) with \( B' \) a finite type \( k \)-subalgebra of \( B \); show \( U \) and \( U' \) have the same image set in \( \text{Spec}(A) \)).

**Extra 3.** *(i)* Let \( X \) be a scheme over an algebraically closed field \( k \). Let \( K/k \) be an extension field. Prove that \( X \) is connected if and only if \( X_K = X \times_k K \) is connected. (hint: study the fibers).

*(ii)* If \( X \) is a connected scheme over a field \( k \) with \( X(k) \) non-empty (e.g., a \( k \)-group scheme) and \( K \) is an extension of \( k \), show that \( X \times_k K \) is connected. In particular, conclude that if \( X \) is a scheme locally of finite type over \( k \) and \( X \times_k L \) is connected for all finite separable extensions \( L/k \), then \( X \times_k K \) is connected for all extensions \( K/k \) (in which case we say that \( X \) is **geometrically connected** over \( k \)).

**Extra 4.** *(i)* Let \( X \) be a group scheme locally of finite type over a field \( k \).

Suppose \( k \) is algebraically closed. Show that \( X_{\text{red}} \) has the structure of a \( k \)-group scheme, and use this to prove \( X_{\text{red}} \) is regular, so the irreducible components of \( X \) coincide with the connected components of \( X \). Using this, prove that if \( X \) is connected, then for non-empty opens \( U, V \) in \( X \), the map \( U \times_k V \to X \) induced by multiplication is surjective. Conclude that if \( X \) is connected then it is necessarily of finite type.

*(ii)* Let \( G \to S \) be a group scheme, \( i : H \to G \) a closed subgroupscheme. We say that \( H \) is **normal** in \( G \) if, for every \( S \)-scheme \( T \), the subgroup \( H(T) \) in \( G(T) \) is a normal subgroup. Define a ‘conjugation’ map \( c_H : H \times_S G \to G \times_S G \) as \((h, g) \mapsto (ghg^{-1} , g)\) via Yoneda, and
show that $H$ is normal in $G$ if and only if $c_H$ factors through the closed sub scheme $H \times_S G$. The map $c_H$ is the ‘universal conjugation’: conjugation of $\text{pr}_2 \in G(H \times_S G)$ by $\text{pr}_1 \in H(H \times_S G)$, followed by using $G(H \times_S G) = G_C(H_G)$, the latter Hom set taken in the category of $G$-schemes, with $(\cdot)_G \overset{\text{def}}{=} (\cdot) \times_S G$, a $G$-scheme via the second projection.

Prove that for any $f : \mathcal{S}' \to \mathcal{S}$, the $\mathcal{S}'$-group scheme $\mathcal{H}' = \mathcal{S}' \times_S \mathcal{H}$ is normal in $\mathcal{G}' = \mathcal{S}' \times_S \mathcal{G}$ when $H$ is normal in $G$, and that the converse holds if $f$ fpqc (actually, only faithful flatness of $f$ is needed). When $k$ is algebraically closed and $G$ and $H$ are reduced and locally of finite type, show $H$ is normal in $G$ iff $H(k)$ is a normal subgroup of $G(k)$.

(iii) Let $k$ be arbitrary. Consider the (necessarily open) connected components of $X$. Explain why the connected component $X_0$ containing the identity point is a closed normal subgroupscheme in $X$. By reducing to the case of algebraically closed $k$, prove that the irreducible components of $X$ are disjoint, and hence are the connected components. Show these components are quasi-compact, and thus are of finite type over $k$.

**Optional Extra 5:** The rest of this assignment is devoted to an additional optional exercise on the operation of “analytification” for algebraic $\mathcal{C}$-schemes with arbitrary singularities; it assumes one has done the exercise on the previous homework concerning analytic spaces. Let $X$ be a scheme locally of finite type over $\mathcal{C}$. Consider the functor on analytic spaces given by $F(Z) = \text{Hom}(Z, X)$, where we take maps in the category of locally ringed spaces of $\mathcal{C}$-algebras. If this is representable, we call a representing object $X^{\text{an}}$ an analytification of $X$ and the functorial isomorphism $F \cong \text{Hom}(\cdot, X^{\text{an}})$ ‘evaluated’ on $X^{\text{an}}$ gives rise (by Yoneda) to a distinguished map $i : X^{\text{an}} \to X$ which is suitably universal for maps from an analytic space to $X$.

(i) Check that if an analytification of $X$ exists, then an analytification of $U$ exists for all open subschemes $U$ in $X$. Conversely, if $\{U_j\}$ is an open covering of $X$ and analytifications of the $U_j$’s exist, show that an analytification of $X$ exists by gluing those for the $U_j$’s.

(ii) If an analytification of $X$ exists, show that one exists for any closed subscheme.

(iii) Show that $\mathcal{C}^n$ serves as an analytification of $\mathcal{A}_\mathcal{C}^n$, with the canonical map $\mathcal{C}^n \to \mathcal{A}_\mathcal{C}^n$ as expected (recall that handout I gave out from EGA concerning maps from a locally ringed space to an affine scheme). Now prove that analytifications exist for any affine $X$, and then for any $X$.

(iv) Note that if $X$ is an affine scheme of finite type over $\mathcal{C}$, then any closed immersion $X \to \mathcal{A}_\mathcal{C}^n$ gives rise to an ‘analytic’ topology on the set $X(C)$ which is independent of the choice of closed immersion and which is finer than the Zariski topology. Use this to directly define an analytic topology on $X(C)$ for any scheme $X$ locally of finite type over $\mathcal{C}$ (this should work if $\mathcal{C}$ is replaced by any topological ring). Now give a ‘choice-free’ definition of the analytification of $X$, with underlying topological space given by $X(C)$ with its ‘analytic’ topology. Using this, explain how $X \rightsquigarrow X^{\text{an}}$ is a functor (noting the effect on underlying sets).

(v) Let $i : X^{\text{an}} \to X$ be an analytification. Prove that for $x \in X(C)$, the local map of local noetherian rings $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\text{an}},x}$ induces an isomorphism on completions and therefore is faithfully flat. In particular, $i$ is a flat map of locally ringed spaces (this was Serre’s motivation for defining flatness!!). If $X \to Z$, $Y \to Z$ are maps of locally finite type
\( \mathbb{C} \)-schemes, describe a natural map \((X \times_Z Y)^{\text{an}} \to X^{\text{an}} \times_Z^{\text{an}} Y^{\text{an}}\). Then prove it is an isomorphism, by building up from the case where \(Z\) is a point, \(X\) and \(Y\) are affine.

It is important to understand the analytification functor. For example, since \(\mathcal{O}_{X^{\text{an}}, x}\) and \(\mathcal{O}_{X, x}\) have isomorphic completions for \(x \in X(\mathbb{C})\), if one of these rings is regular then so is the other; thus, \(X\) is non-singular at \(x\) if and only if \(X^{\text{an}}\) is a complex manifold near \(x\)! Also, it can be shown that \(X\) connected in the Zariski topology if and only if \(X^{\text{an}}\) is connected, \(X\) is separated (resp. proper) over \(\text{Spec}(\mathbb{C})\) if and only if \(X^{\text{an}}\) is Hausdorff (resp. compact Hausdorff), etc. Grothendieck and Serre proved that the functor \(X \to X^{\text{an}}\) between proper \(\mathbb{C}\)-schemes and compact Hausdorff analytic spaces is fully faithful.