Math 216A. Sheafification

1. Introduction

Let X be a topological space, and $\mathscr{F}$ a presheaf on X. This handout gives some additional details on the construction of sheafification provided in class. The construction we discussed is totally different from the one given in [H]. The construction in [H, Ch. II, Prop. 1.2] is certainly *much* simpler. But the reason we prefer the alternative construction as in class is because it is the approach one must use when adapting the idea of sheafification to the broader context of Grothendieck topologies later in life (e.g., the famous "étale topology" developed in SGA4 which is the key to the solution of the Weil Conjectures and underlies a vast amount of modern algberaic geometry and number theory, as well as the process of "stackification" used to make sense of "quotient stacks" in the theory of Artin stacks that provides a powerful generalization of the theory of schemes for advanced work with moduli problems).

In class we defined a partial order "$\geq$" via refinement on the set Σ_U of open covers $\mathcal{V} = \{V_i\}_{i\in I}$ having no repetitions (i.e., $V_i \neq V_j$ for $i \neq j$); this "no repetitions" condition ensures Σ_U is a set. We defined

$$\mathscr{F}_0(U) = \varinjlim_{\mathcal{V}\in\Sigma_U} D(\mathcal{V})$$

where

$$D(\mathcal{V}) = \{(s_i) \in \prod_{i\in I} \mathscr{F}(V_i) \mid s_i|_{V_{ij}} = s_j|_{V_{ij}} \text{ in } \mathscr{F}(V_i \cap V_j)\}$$

is the set of "compatible local data relative to $\mathcal{V}$" and these constitute a directed system. More specifically, if $\{V'_{i'}\}_{i'\in I'} =: \mathcal{V}' \geq \mathcal{V}; = \{V_i\}_{i\in I}$ with $\tau : I' \to I$ a "refinement" map for the index sets (i.e,, $V'_{i'} \subset V_{\tau(i')}$ for all $i' \in I'$) then we define the transition map $D(\mathcal{V}) \to D(\mathcal{V}')$ via $(s_i) \mapsto (s_{\tau(i')}|_{V_{i'}})$. This transition map is *independent* of τ because if $\sigma : I' \to I$ is also a "refinement" map then $V'_{i'} \subset V_{\tau(i')}, V_{\sigma(i')}$ implies $V'_{i'} \subset V_{\tau(i')} \cap V_{\sigma(i')}$, so the agreement of $s_{\tau(i')}$ and $s_{\sigma(i')}$ upon restriction over $V_{\tau(i')} \cap V_{\sigma(i')}$ implies that their restrictions to $V'_{i'}$ coincide. Such independence of the choice of τ is the reason that the direct limit defining $\mathscr{F}_0(U)$ is really intrinsic to $\mathscr{F}$ and U (not reliant upon auxiliary choices).

2. Basic constructions

Having defined $\mathscr{F}_0(U)$ for open $U \subset X$, we have an evident map of sets $\mathscr{F}(U) \to \mathscr{F}_0(U)$ via $s \mapsto s \in D(\{U\})$ (the image of s in each $D(\mathcal{V})$ is $(s|_{V_i})$). If $U' \subset U$ is an open subset then we define $\Sigma_U \to \Sigma_{U'}$ compatible with "$\geq$" via $\mathcal{V} \mapsto \mathcal{V} \cap U' := \{V_i \cap U'\}$ (dropping repetitions), and the restriction map $\mathscr{F}_0(U) \to \mathscr{F}_0(U')$ comes from the maps $D(\mathcal{V}) \to D(\mathcal{V} \cap U')$ defined by $(s_i) \mapsto (s_i|_{V_i \cap U'})$. It is then easy to check (do it) that these restriction maps $\mathscr{F}_0(U) \to \mathscr{F}_0(U')$ satisfy the transitivity condition to make $\mathscr{F}_0$ a presheaf, and that the maps $\mathscr{F}(U) \to \mathscr{F}_0(U)$ are compatible with these restrictions, so we thereby obtain a *map of presheaves*

$$\theta_0 : \mathscr{F} \to \mathscr{F}_0.$$

Lemma 2.1. *If $\mathscr{F}$ is a sheaf then θ_0 is an isomorphism.*

Proof. The sheaf axiom gives that for any $\mathcal{V} \in \Sigma_U$, the natural map $\mathscr{F}(U) = D(\{U\}) \to D(\mathcal{V})$ is an isomorphism. This latter map is compatible with the transition maps $D(\mathcal{V}) \to D(\mathcal{V}')$ for $\mathcal{V}' \geq \mathcal{V}$, so passing to the direct limit over such $\mathcal{V}$'s yields that the natural map $\theta_0 : \mathscr{F}(U) \to \mathscr{F}_0(U)$ is an isomorphism for all U, so we are done. ∎

Now we show that θ_0 satisfies two of the desired properties of sheafification: the stalk condition and a suitable version of the universal mapping property despite that maybe the presheaf $\mathscr{F}_0$ isn't a sheaf!

Lemma 2.2. *For all $x \in X$, the map on stalks $(\theta_0)_x : \mathscr{F}_x \to (\mathscr{F}_0)_x$ is an isomorphism.*

Proof. Pick $x \in X$. Any germ $\xi \in (\mathscr{F}_0)_x$ arises from $\mathscr{F}_0(U)$ for some open U around x, so it is represented by compatible local data $(s_i) \in \prod \mathscr{F}(V_i)$ for some open cover $\{V_i\}$ of U. Thus, x lies in some V_{i_0}, and the compatibility among the s_i's implies that the image of (s_i) under $D(\mathcal{V}) \to D(\mathcal{V} \cap V_{i_0})$ arises from $s_{i_0} \in \mathscr{F}(V_{i_0}) = D(\{V_{i_0}\})$. In other words, $\theta_0 : \mathscr{F}(V_{i_0}) \to \mathscr{F}_0(V_{i_0})$ carries s_{i_0} to a representative of ξ. This shows that $(\theta_0)_x$ is surjective for all x.

Suppose germs $\xi, \xi' \in \mathscr{F}_x$ are carried to the same place under $(\theta_0)_x$, so for a sufficiently small open $U \subset X$ around x we can pick representatives $s, s' \in \mathscr{F}(U)$ so that $\theta_0(s) = \theta_0(s')$ in $\mathscr{F}_0(U)$. The definition of $\mathscr{F}_0(U)$ as a direct limit provides an open cover $\mathcal{V} = \{V_i\}$ of U so that $(s|_{V_i}) = (s|_{V_i})$ in $\prod \mathscr{F}(V_i)$. In other words, $s|_{V_i} = s'|_{V_i}$ for all $i \in I$. In particular, since $\{V_i\}$ is an open cover of the open set U contaning x, some V_{i_0} contains x. Thus, the equality $s|_{V_{i_0}} = s'|_{V_{i_0}}$ in $\mathscr{F}(V_{i_0})$ implies that the associated germs $s_x, s'_x \in \mathscr{F}_x$ coincide. But by design of s and s' we have $s_x = \xi$ and $s'_x = \xi'$, so $\xi = \xi'$ as desired. ∎

Proposition 2.3. *If $f : \mathscr{F} \to \mathscr{G}$ is a map to a sheaf then it uniquely factors through θ_0 (i.e., there is a unique $h : \mathscr{F}_0 \to \mathscr{G}$ so that $h \circ \theta_0 = f$).*

Proof. The construction of $\mathscr{F}_0$ is "functorial in $\mathscr{F}$", meaning that there is an evident associated map $f_0 : \mathscr{F}_0 \to \mathscr{G}_0$ (arising on sections over any open $U \subset X$ via passage to the direct limit over Σ_U of the maps $D_{\mathscr{F}}(\mathcal{V}) \to D_{\mathscr{G}}(\mathcal{V})$ induced by $\prod \mathscr{F}(V_i) \to \prod \mathscr{G}(V_i)$ defined by f factorwise) making the diagram of presheaves

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\theta_{0,\mathscr{F}}"'] & \mathscr{G} \arrow[d, "\theta_{0,\mathscr{G}}", "\simeq"'] \\ \mathscr{F}_0 \arrow[r, "f_0"'] & \mathscr{G}_0 \end{tikzcd}$$

commute; the right side is an isomorphism because $\mathscr{G}$ is a sheaf. The composition $h = \theta_{0,\mathscr{G}}^{-1} \circ f_0$ satisfies

$$h \circ \theta_{0,\mathscr{F}} = \theta_{0,\mathscr{G}}^{-1} \circ f_0 \circ \theta_{0,\mathscr{F}} = \theta_{0,\mathscr{G}}^{-1} \circ \theta_{0,\mathscr{G}} \circ f = f.$$

We have built the desired factorization of f through $\theta_0 = \theta_{0,\mathscr{F}}$, and it remains to show that such a factorization is *unique*.

Say $h' : \mathscr{F}_0 \rightrightarrows \mathscr{G}$ is a map of presheaves satisfying $h' \circ \theta_0 = f$. We want to show that $h' = h$ for h as above. We have shown that $h \circ \theta_0 = f$, so passing to the stalks at each $x \in X$

gives an equality of stalk maps $h_x \circ (\theta_0)_x = f_x = h'_x \circ (\theta_0)_x$ with $(\theta_0)_x$ an isomorphism, so $h_x = h'_x$ as maps $\mathscr{F}_x \rightrightarrows \mathscr{G}_x$. Hence, for any open $U \subset X$ and $s \in \mathscr{F}(U)$ we have

$$h(s)_x = h_x(s_x) = h'_x(s_x) = h'(s)_x.$$

Thus, the elements $h(s), h'(s) \in \mathscr{G}(U)$ that we want to show are equal (yielding $h = h'$) have the same stalk at each $x \in U$, so they have the same restriction to some open $U_x \subset U$ around x. Such opens U_x constitute an open cover of U, so since $\mathscr{G}$ is a *sheaf* we conclude that $h(s) = h'(s)$ as desired. ∎

Putting this all together, *if* $\mathscr{F}_0$ were a sheaf then $(\mathscr{F}_0, \theta_0)$ would satisfy the desired properties. Alas, this may not be a sheaf. To overcome this issue, we next investigate more closely the properties of $\mathscr{F}_0$.

3. Separated presheaves

By definition, a sheaf is a presheaf $\mathscr{F}$ satisfying two conditions: local uniqueness (i.e., $s, t \in \mathscr{F}(U)$ agreeing upon restriction to members of an open cover of U must satisfy $s = t$) and gluing of compatible local data. If $\mathscr{F}$ satisfies just the first of these two conditions, we call it a *separated* presheaf. (This is a notion that only ever arises essentially just in the construction of sheafification.) Though $\mathscr{F}_0$ may not be a sheaf, it gets us at least halfway there:

Lemma 3.1. *For any presheaf $\mathscr{F}$, the presheaf $\mathscr{F}_0$ is separated.*

Proof. Say $s, t \in \mathscr{F}_0(U)$ for open $U \subset X$ satisfy $s|_{U_i} = t|_{U_i}$ in $\mathscr{F}_0(U_i)$ for all members U_i of an open cover of U. We want to conclude that $s = t$. Since Σ_U is a directed set relative to refinement, we can pick an open cover $\mathcal{V} = \{V_j\}$ of U so that s, t both arise from $D(\mathcal{V})$. That is, s and t respectively arise from $(s_j), (t_j) \in \prod \mathscr{F}(V_j)$ which satisfy overlap compatibility. For an open cover $\mathcal{V}'$ of U refining both $\mathcal{V}$ and $\{U_i\}$, one checks that $D(\mathcal{V}) \to D(\mathcal{V}')$ carries (s_j) and (t_j) to the same place. Hence, in the direct limit $\mathscr{F}_0(U)$, the images of s and t coincide as desired. ∎

In view of the preceding lemma, iterating the construction $\mathscr{F} \rightsquigarrow (\mathscr{F}_0, \theta_0)$ *twice* does the job once we show:

Lemma 3.2. *If $\mathscr{F}$ is a separated presheaf then $\mathscr{F}_0$ is a sheaf.*

Proof. We already know that $\mathscr{F}_0$ is always a separated presheaf, so the issue is to check that it satisfies the gluing axiom when $\mathscr{F}$ is separated. That is, if U is an open subset of X and $\mathcal{V} = \{V_i\}$ is an open cover of U then for any $(s_i) \in \prod \mathscr{F}_0(V_i)$ satisfying $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ in $\mathscr{F}_0(V_i \cap V_j)$ for all i, j we seek $s \in \mathscr{F}_0(U)$ such that $s|_{V_i} = s_i$ in $\mathscr{F}_0(V_i)$ for all i.

It is enough to find $s \in \mathscr{F}_0(U)$ and open covers $\{V_{i,\alpha}\}_{\alpha \in A_i}$ of each V_i such that $s|_{V_{i,\alpha}} = a_i|_{V_{i,\alpha}}$ in $\mathscr{F}_0(V_{i,\alpha})$ for all $\alpha \in A_i$. Indeed, then separatedness of $\mathscr{F}_0$ (!) would force the sections $s|_{V_i}, s_i \in \mathscr{F}_0(V_i)$ to coincide for each i, as desired.

By definition, each $s_i \in \mathscr{F}_0(V_i)$ comes from $(s_{i,\alpha}) \in \prod_{\alpha \in A_i} \mathscr{F}(V_{i,\alpha})$ for some open cover $\{V_{i,\alpha}\}_{\alpha \in A_i}$ of V_i. The hypothesis that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ in the direct limit $\mathscr{F}_0(V_i \cap V_j)$ over the index set $\Sigma_{V_i \cap V_j}$ directed by refinement provides $\mathcal{V}$ refining the open cover

$$\{V_{i,\alpha} \cap V_{j,\beta}\}_{(\alpha,\beta) \in A_i \times A_j}$$

of $V_i \cap V_j$ such that each $s_{i,\alpha}$ and $s_{j,\beta}$ have the same restriction into $\mathscr{F}(V)$ for all $V \in \mathcal{V}$. But $\mathscr{F}$ is *separated* by hypothesis, so the agreement of restrictions over constituents V of an open cover of $V_{i,\alpha} \cap V_{j,\beta}$ forces actual equality

$$s_{i,\alpha}|_{V_{i,\alpha} \cap V_{j,\beta}} = s_{j,\beta}|_{V_{i,\alpha} \cap V_{j,\beta}}$$

in $\mathscr{F}(V_{i,\alpha} \cap V_{j,\beta})$! Hence, the element

$$(s_{i,\alpha}) \in \prod \mathscr{F}(V_{i,\alpha})$$

satisfies the overlap compatibility condition to belong to $D(\{V_{i,\alpha}\})$, so it represents an element s in the direct limit in $\mathscr{F}_0(U)$. By design, $s|_{V_{i,\alpha}} = s_i|_{V_{i,\alpha}}$ in $\mathscr{F}_0(V_{i,\alpha})$ for all $i \in I$ and $\alpha \in A_i$. Hence, as we saw already via the separatedness of $\mathscr{F}_0$, this s does the job! $\blacksquare$