Math 216A. Extensions of Quasi-coherent Sheaves

1. Main result

For modules M and N over a ring A, an *extension* of M by N is a short exact sequence of A-modules

$$0 \to N \to E \to M \to 0.$$

Likewise, for sheaves of modules $\mathscr{F}'$ and $\mathscr{F}''$ on a ringed space $(X, \mathscr{O}_X)$, an *extension* of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0. \tag{1}$$

The aim of this handout is to discuss how quasi-coherence behaves under the formation of extensions. We will prove:

Proposition 1.1. *For any scheme X and short exact sequence* (1) *of $\mathscr{O}_X$-modules, if $\mathscr{F}'$ and $\mathscr{F}''$ are quasi-coherent then so is $\mathscr{F}$.*

This assertion is of local nature on X, so we can assume $X = \operatorname{Spec} A$ is affine. By the Localization Criterion for quasi-coherence on affine schemes, to show $\mathscr{F}$ is quasi-coherent we just need to check that for all $a \in A$ the natural map

$$\theta_{\mathscr{F},a} : \mathscr{F}(X)_a \to \mathscr{F}(D(a))$$

is an isomorphism. The analogous maps for $\mathscr{F}'$ and $\mathscr{F}''$ are isomorphisms since these sheaves are assumed to be quasi-coherent. Moreover, by naturality of $\theta_{\mathscr{F},a}$ in $\mathscr{F}$ combined with the left-exactness of the formation of sections over an open subset and the exactness of localization at a we have a commutative diagram of left-exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(X)_a \arrow[r] \arrow[d, "\simeq"'] & \mathscr{F}(X)_a \arrow[r] \arrow[d] & \mathscr{F}''(X)_a \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathscr{F}'(D(a)) \arrow[r] & \mathscr{F}(D(a)) \arrow[r] & \mathscr{F}''(D(a))
\end{tikzcd}$$

Due to the indicated vertical isomorphisms, a simple diagram chase shows that the middle arrow (which we want to be an isomorphism) is at least injective. Moreover, if we had surjectivity at the right end along the top then a diagram chase would yield surjectivity for the middle arrow too, and so we would be done. Thus, it suffices to show that $\mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective (as then its a-localization is surjective for any $a \in A$).

In general, the formation of global sections is merely left-exact and *not* right-exact. (The vast edifice of sheaf cohomology is all about the failure of such right-exactness.) So how could we reasonably expect to prove $\mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective? What special feature of our situation could lead to such a hope being realized? It is that the kernel $\mathscr{F}' = \ker(\mathscr{F} \twoheadrightarrow \mathscr{F}'')$ is quasi-coherent. More generally:

Theorem 1.2. *For any short exact sequence* (1) *on an affine scheme $X = \operatorname{Spec}(A)$ with $\mathscr{F}'$ quasi-coherent, the map $\mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective.*

Our proof of this below is a reinterpretation of the proof given in [H, Ch. II, Prop. 5.6], expressed in the more conceptual language of "cohomology", though we will not actually require any knowledge of what sheaf cohomology is. Nonetheless, the approach we take is a good first warm-up to a style of analysis that appears over and over again in the study of sheaf cohomology in Math 216B and beyond.

Remark 1.3. Although the length of our proof below is somewhat longer than for [H, Ch. II, Prop. 5.6] (which itself relies on [H, Ch. II, Lemma 5.3] that is exactly a bare-hands formulation of the localization property $\mathscr{G}(\mathrm{Spec}(A))_a \simeq \mathscr{G}(\mathrm{Spec}(A_a))$ for a quasi-coherent sheaf on an affine scheme), the main content of the computational work is really the same. This is because the "partition of unity" step in the proof of [H, Ch. II, Prop. 5.6] is lurking inside the fact that the "$\mathscr{B}$-presheaf" in the construction of $\widetilde{M}$ satisfies the $\mathscr{B}$-sheaf property to ensure the associated sheaf has the expected module of global sections over an affine scheme. The main merit of our argument below is that it puts the computations into a broader conceptual context which will be investigated extensively in Math 216B.

2. A reformulation

Let's explain how Theorem 1.2 can be reduced to a problem entirely about $\mathscr{F}'$, making essentially *no reference* to $\mathscr{F}$ or $\mathscr{F}''$. This will have nothing to do with schemes or quasi-coherence, so for the moment we shall work with an arbitrary topological space X and a short exact sequence of abelian sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Pick $s \in \mathscr{F}''(X)$. We want to formulate an "obstruction" to being in the image of $\mathscr{F}(X)$. Locally on X, the section s lifts to a section of $\mathscr{F}$ (a general feature of sheaf surjections). Thus, there is an open cover $\{U_j\}_{j \in J}$ of X such that $s|_{U_j}$ lifts to some $t_j \in \mathscr{F}(U_j)$ for each $j \in J$. Keep in mind that this choice of open cover *depends on s*.

If the t_j's were to agree on overlaps then they would glue to a section $t \in \mathscr{F}(X)$ which does the job: $t \mapsto s$ in $\mathscr{F}''(X)$ since this can be checked over each U_j (where it holds since $t|_{U_j} = t_j$ in $\mathscr{F}(U_j)$ lifts $s|_{U_j} \in \mathscr{F}''(U_j)$ by design). But typically there is no reason for the t_j's to agree on overlaps, since there is so much flexibility in how each is chosen: there is the freedom to change t_j by an element $t'_j \in \ker(\mathscr{F}(U_j) \to \mathscr{F}''(U_j)) = \mathscr{F}'(U_j)$ for each j.

Letting $U_{jj'} := U_j \cap U_{j'}$, the difference $t_{j'} - t_j \in \mathscr{F}(U_{jj'})$ has image in $\mathscr{F}''(U_{jj'})$ equal to $(s|_{U_{j'}})|_{U_{jj'}} - (s|_{U_j})|_{U_{jj'}} = s|_{U_{jj'}} - s|_{U_{jj'}} = 0$, so $t_{j'} - t_j$ comes from an element

$$t_{jj'} \in \ker(\mathscr{F}(U_{jj'}) \to \mathscr{F}''(U_{jj'})) = \mathscr{F}'(U_{jj'}).$$

These elements in the $\mathscr{F}'(U_{jj"})$'s are not completely arbitrary: they satisfy the "cocycle condition" over the triple overlaps $U_{jj'j''}$ (omitting "restrict to $U_{jj'j''}$" notation):

$$t_{jj'} - t_{jj''} + t_{j'j''} = (t_{j'} - t_j) - (t_{j''} - t_j) + (t_{j''} - t_{j'}) = 0$$

in $\mathscr{F}'(U_{jj'j''})$.

If we were to replace t_j with $t_j + \tau_j$ for $\tau_j \in \mathscr{F}'(U_j)$ (the maximal flexibility we have) then $t_{jj'}$ is replaced with $t_{jj'} + (\tau_{j'}|_{U_{jj'}} - \tau_j|_{U_{jj'}})$. Thus, we seek such elements τ_j which make the modified $t_{jj'}$'s all vanish. In other words, we seek elements $\tau_j \in \mathscr{F}'(U_j)$ for all j so that $t_{jj'} = \tau_j|_{U_{jj"}} - \tau_{j'}|_{U_{jj'}}$ in $\mathscr{F}'(U_{jj'})$ for all j, j'.

Our problem has been reduced to one entirely about $\mathscr{F}'$ *essentially without any reference* to $\mathscr{F}$ and $\mathscr{F}''$ as follows. Let $\mathfrak{U} = \{U_j\}_{j\in J}$ be an open cover of X, and define the group $Z^1(\mathfrak{U}, \mathscr{F}')$ of *Čech* 1-*cocycles* for $\mathscr{F}'$ relative to $\mathfrak{U}$ to consist of tuples

$$t = (t_{jj'}) \in \prod_{(j,j')} \mathscr{F}'(U_{jj'})$$

satisfying the "cocycle condition" $t_{jj'} - t_{jj''} + t_{j'j''} = 0$ in $\mathscr{F}'(U_{jj'j''})$ for all $j, j', j'' \in J$. There is the subgroup $B^1(\mathfrak{U}, \mathscr{F}')$ of "trivial" 1-cocycles (called 1-*coboundaries*) consisting of tuples $(t_{jj'})_{(j,j')}$ for which $t_{jj'} = \tau_j|_{U_{jj"}} - \tau_{j'}|_{U_{jj'}}$ for an element $\tau = (\tau_j) \in \prod_j \mathscr{F}'(U_j)$.

The preceding arguments showed that for any $s \in \mathscr{F}''(X)$ such that $s|_{U_j}$ lifts to $\mathscr{F}(U_j)$ for all $j \in J$, there is an associated 1-cocycle in $Z^1(\mathfrak{U}, \mathscr{F}')$ well-defined modulo the subgroup $B^1(\mathfrak{U}, \mathscr{F}')$ of 1-coboundaries, and that s comes from $\mathscr{F}(X)$ if and only if the 1-cocycle we've made is a 1-coboundary. In other words, to such an s we have built a canonical element

$$[s] \in \mathrm{H}^1(\mathfrak{U}, \mathscr{F}') := Z^1(\mathfrak{U}, \mathscr{F}')/B^1(\mathfrak{U}, \mathscr{F}')$$

whose vanishing is necessary and sufficient for s to lift to $\mathscr{F}(X)$. In §3 we will show that if X is affine, $\mathscr{F}'$ is quasi-coherent, and $\mathfrak{U}$ is a finite open cover by basic affine opens then $[s] = 0$. That will then complete the proof of Theorem 1.2.

The aim of showing the vanishing of the specific element $[s]$ of course makes extensive reference to $\mathscr{F}''$ and $\mathscr{F}$ (and the specific $\mathfrak{U}$ being considered was built from choices involving s too), so this does not really make everything reduce to a problem intrinsic to $\mathscr{F}'$. But in Math 216B, you will learn that when X is affine, $\mathscr{F}'$ is quasi-coherent, and all U_j are affine then the entire group $\mathrm{H}^1(\mathfrak{U}, \mathscr{F}')$ vanishes. That is much more than we need, and this much stronger vanishing *is* intrinsic to $\mathscr{F}'$ (and the choice of $\mathfrak{U}$, which can be arbitrary for the purpose of defining $\mathrm{H}^1(\mathfrak{U}, \mathscr{F}')$). It is also a step towards a deeper vanishing result for positive-degree sheaf cohomology of quasi-coherent sheaves on affine schemes. Such wider general vanishing lies much beyond what we need here. Nontheless, that gives some broader context to "why" the vanishing we will prove below is true and more importantly can be regarded as an instance of a deeper vanishing property that really is intrinsic to $\mathscr{F}'$.

Remark 2.1. There is a very useful property of $[s]$ in the above generality (that will also help us in the proof of its vanishing in the special case we need): if $V \subset X$ is an open subset then for the associated open cover $V \cap \mathfrak{U} = \{V \cap U_j\}_{j\in J}$ we have a corresponding element $[s|_V] \in \mathrm{H}^1(V \cap \mathfrak{U}, \mathscr{F}'|_V)$ and there is an obvious "restriction" map

$$\mathrm{H}^1(\mathfrak{U}, \mathscr{F}') \to \mathrm{H}^1(V \cap \mathfrak{U}, \mathscr{F}'|_V).$$

The useful property is that this latter map carries $[s]$ to $[s|_V]$, the verification of which is left as an exercise in inspecting the definitions.

3. A localization trick

Now we focus on a special case: $X = \mathrm{Spec}(A)$ is affine, $\mathscr{F}'$ is quasi-coherent, and $\mathfrak{U} = \{U_1, \dots, U_n\}$ is a finite open cover of X by basic affine opens $U_j = D(a_j) = \mathrm{Spec}(A_{a_j})$. By Remark 2.1, for any $U_j \in \mathfrak{U}$ the natural map

$$\rho_j : \mathrm{H}^1(\mathfrak{U}, \mathscr{F}') \to \mathrm{H}^1(U_j \cap \mathfrak{U}, \mathscr{F}'|_{U_j})$$

carries $[s]$ to $[s|_{U_j}]$, but we chose each U_j so that $s|_{U_j}$ lifts to $\mathscr{F}(U_j)$ and hence $[s|_{U_j}] = 0$. In other words, $\rho_j([s]) = 0$ for all j. To exploit this, we finally make use of the quasi-coherence of $\mathscr{F}'$:

Lemma 3.1. *For any basic affine open $U = D(a) \subset \mathrm{Spec}(A) = X$, the natural map*

$$\mathrm{H}^1(\mathfrak{U}, \mathscr{F}')_a \to \mathrm{H}^1(U \cap \mathfrak{U}, \mathscr{F}'|_U)$$

(induced by the "restriction" $\mathrm{H}^1(\mathfrak{U}, \mathscr{F}') \to \mathrm{H}^1(U \cap \mathfrak{U}, \mathscr{F}'|_U)$ that is linear over $A \to A_a$) is an isomorphism.

Proof. There are compatible restriction maps

$$Z^1(\mathfrak{U}, \mathscr{F}') \to Z^1(U \cap \mathfrak{U}, \mathscr{F}'|_U), \;\; B^1(\mathfrak{U}, \mathscr{F}') \to B^1(U \cap \mathfrak{U}, \mathscr{F}'|_U)$$

linear over $A \to A_a$, so it suffices to show that the resulting A_a-linear maps

$$Z^1(\mathfrak{U}, \mathscr{F}')_a \to Z^1(U \cap \mathfrak{U}, \mathscr{F}'|_U), \;\; B^1(\mathfrak{U}, \mathscr{F}')_a \to B^1(U \cap \mathfrak{U}, \mathscr{F}'|_U)$$

are isomorphisms.

But if one goes back to how Z^1 and B^1 are defined, one sees that both source and target of each map are defined in terms of compatible kernels and images of A-linear maps built in terms of compatible *finite direct products* among $\mathscr{F}'(U_j)$, $\mathscr{F}'(U_{jj'})$, $\mathscr{F}'(U_{jj'j''})$, and analogues with these open sets replacing with their intersection with $U = D(a)$. Since a-localization commutes with finite direct products, the maps in question all naturally arise from instances of the maps

$$\mathscr{F}'(V)_a \to \mathscr{F}'(D(a) \cap V) \tag{2}$$

for basic affine open $V = \mathrm{Spec}(B) \subset \mathrm{Spec}(A)$ and so it suffices (check!) to show that the latter maps are all isomorphisms. If $A \to B$ carries a to an element $b \in B$ then $D(a) \cap V = D(b)$, and so for the quasi-coherent restriction $\mathscr{G} = \mathscr{F}'|_V$ on $V = \mathrm{Spec}(B)$ the map (2) is identified with the natural map $\mathscr{G}(V)_b \to \mathscr{G}(D(b))$. But this latter map is an isomorphism due to quasi-coherence! ■

This Lemma applied to $U = U_j = D(a_j)$ leads us to the conclusion that the element $[s] \in \mathrm{H}^1(\mathfrak{U}, \mathscr{F}')$ has vanishing image in the localization $\mathrm{H}^1(\mathfrak{U}, \mathscr{F}')_{a_j}$ for every j. But for any A-module M (e.g., $M = \mathrm{H}^1(\mathfrak{U}, \mathscr{F}')$) an element $m \in M$ with vanishing image in every M_{a_j} is equal to 0: this is either seen via bare hands with stalks at primes, or by reinterpreting the map $M \to \prod_j M_{a_j}$ as $\widetilde{M}(X) \to \prod_j \widetilde{M}(U_j)$ that is injective by the sheaf property of $\widetilde{M}$ (since the U_j's constitute an open cover of X). Thus, $[s] = 0$ as desired!