1. SOME BASIC DEFINITIONS

Let $S = \oplus_{n \geq 0} S_n$ be an $\mathbb{N}$-graded ring (we follows French terminology here, even though outside of France it is commonly accepted that $\mathbb{N}$ does not include 0). Morphisms between $\mathbb{N}$-graded rings are understood to respect the grading. The irrelevant ideal is

$$S_+ = \oplus_{n > 0} S_n;$$

keep in mind that we allow $S_0$ to have a nontrivial ideal theory (that is, it need not be a field). An element $f \in S$ is homogeneous if $f \in S_d$ for some $d$, and then $d$ is unique if $f \neq 0$; we call $d$ the degree of $f$ (when $f \neq 0$), and we consider 0 as having arbitrary degree. Note that the equation

$$\deg(fg) = \deg(f) + \deg g$$

is valid even if one of $f$, $g$, or $fg$ vanishes, using the convention that 0 may be considered to have arbitrary degree. For example, $S_+$ is exactly the set of elements (including 0) with positive degree.

For a general element $f \in S$, the homogeneous parts of $f$ are the projections $f_d$ of $f$ into each $S_d$ (so $f_d = 0$ for all but finitely many $d$).

An ideal $I$ in $S$ is homogeneous if an element $f = \sum n \geq 0 f_n$ of $S$ lies in $I$ if and only if each homogeneous part $f_n$ lies in $I$. It is a simple exercise (inducting on degrees) to check that an ideal generated by homogeneous elements is a homogeneous ideal, and that homogeneous ideal $I$ in $S$ is prime if and only if it is a proper ideal and

$$fg \in I \Rightarrow f \in I \text{ or } g \in I$$

for homogeneous $f, g \in S$. It is also clear that the kernel of a morphism of $\mathbb{N}$-graded rings is a homogeneous ideal, and that for any homogeneous ideal $I$ of $S$ there is a natural $\mathbb{N}$-grading on $S/I$.

**Definition 1.1.** Let $S$ be an $\mathbb{N}$-graded ring. The topological space $\text{Proj}(S)$ has underlying set

$$\text{Proj}(S) = \{ p \text{ a homogeneous prime such that } S_+ \nsubseteq p \},$$

and the closed sets are the loci $V(I) = \{ p \in \text{Proj}(S) | I \subseteq p \}$ for homogeneous ideals $I$ of $S$ (context will prevent confusion with the analogous “$V(I)$” notation for affine schemes).

It is easy to check that the $V(I)$’s do satisfy the axioms to define the closed sets for a topology on $\text{Proj}(S)$ (the empty set is $V(S)$ and $\text{Proj}(S) = V(0)$). A homogeneous prime $p$ fails to contain $I$ if and only if there exists a homogeneous element $f \in I$ that does not lie in $p$ (here we use crucially that $I$ is homogeneous).

Thus, a base of open sets for the topology on $\text{Proj}(S)$ is given by loci

$$D_+(f) = \{ p \in \text{Proj}(S) | f \not\in p \} = \text{Proj}(S) - V(fS)$$

for homogeneous $f \in S$. A crucial fact is that it is even enough to take $f$ with positive degree:

**Lemma 1.2.** A base of open sets for the topology on $\text{Proj}(S)$ is given by loci $D_+(f)$ for homogeneous $f \in S_+$.

**Proof.** Consider a homogeneous $f \in S$ and a point $p \in D_+(f)$. We need to find a homogeneous element $g \in S_+$ such that $p \in D_+(g) \subseteq D_+(f)$. Since $p$ does not contain $S_+$ (by the definition of $\text{Proj}(S)$), there exists $h \in S_+$ not in $p$. Thus, the condition $f \not\in p$ implies $fh \not\in p$, and $fh$ is homogenous with positive degree (possibly even $fh = 0$) since both $f$ and $h$ are homogeneous and $\deg h > 0$. We conclude that $p \in D_+(fh)$, and clearly $D_+(fh) \subseteq D_+(f)$. 

**Beware** that $\text{Proj}(S)$ is generally not quasi-compact! For example, $\text{Proj}(k[x_1, x_2, \ldots])$ with infinitely many indeterminates of degree 1 is not quasi-compact, as it is covered by opens $D_+(x_i)$ and there is evidently no finite subcover. This issue is best understood as follows:
Theorem 1.3. For an \( \mathbb{N} \)-graded ring \( S \), \( \text{Proj}(S) \) is empty if and only if all elements of \( S_+ \) are nilpotent. More generally, for positive-degree homogeneous elements \( f \) and \( \{f_i\}_{i \in I} \) in \( S \), \( D_+(f) \subseteq \cup D_+(f_i) \) if and only if some power of \( f \) lies in the homogeneous ideal generated by the \( f_i \)'s.

In particular, a collection of \( D_+(f_i) \)'s with all deg \( f_i > 0 \) covers \( \text{Proj}(S) \) if and only if every element of \( S_+ \) has some power lying in the homogeneous ideal generated by the \( f_i \)'s.

The contrast with \( \text{Spec} \) is of course that \( \text{Spec}(A_{f_i}) \)'s cover \( \text{Spec} A \) if and only if the \( f_i \)'s generate the unit ideal. The interference of \( S_+ \) in the analogous covering criterion for \( \text{Proj} \), coupled with the possibility that \( S_+ \) might not be finitely generated, is the reason why \( \text{Proj}(S) \) can fail to be quasi-compact. On the other hand, in most interesting situations the ideal \( S_+ \) is finitely generated and hence \( \text{Proj}(S) \) is quasi-compact. However, we note that some fundamental constructions of Mumford in the study of moduli of abelian varieties rest crucially on the use of non-quasi-compact \( \text{Proj} \)'s.

**Proof.** Let \( I \) be the homogeneous ideal generated by the \( f_i \)'s, so the complement of \( \cup D_+(f_i) \) is the set of \( p \in \text{Proj}(S) \) that contain \( I \). Hence, we need to determine when \( D_+(f) \) is disjoint from the set of such \( p \)'s, or equivalently when every \( p \in \text{Proj}(S) \) that contains \( I \) also contains \( f \); we want to show that this condition is exactly the condition that a power of \( f \) lies in \( I \). Passing to the \( \mathbb{N} \)-graded \( S/I \), we are reduced to proving that a homogeneous \( f \in S_+ \) lies in \( p \) for all \( p \in \text{Proj}(S) \) if and only if \( f \) is nilpotent; keep in mind that \( f \) has positive degree. One direction is obvious, and conversely we must prove that if \( f \in S_+ \) is homogeneous of degree \( d > 0 \) and \( f \) is not nilpotent, then there exists a homogeneous prime \( p \) such that \( f \not\subseteq p \) (and so \( S_+ \not\subseteq p \) too, so \( p \in \text{Proj}(S) \)).

We will make use of an auxiliary construction that will play an important role later. Let \( S^{(d)} = \bigoplus_{n \geq 0} S_{dn} \) (so \( S^{(d)} = S \) if \( d = 1 \)). This is naturally an \( \mathbb{N} \)-graded ring with vanishing graded pieces in degrees not divisible by \( d \). Consider the localized ring \( (S^{(d)})_f \); since \( (S^{(d)})_f = S^{(d)}[T]/(1 - Tf) \), by assigning \( T \) degree \( -d \) we see that \( (S^{(d)})_f \) naturally has a \( \mathbb{Z} \)-grading (with vanishing terms away from degrees divisible by \( d \)). For example, \( s/f^n \) is assigned degree \( d \) for homogeneous elements \( s \in S^{(d)} \).

Let \( (S^{(d)})_{(f)} \subseteq (S^{(d)})_f \) denote the direct summand of degree-0 elements in the \( \mathbb{Z} \)-graded \( (S^{(d)})_f \). This is a ring, and if \( f \) is not nilpotent in \( S \) then it is not nilpotent in \( S^{(d)} \), so then \( (S^{(d)})_f \neq 0 \) and hence the subring \( (S^{(d)})_{(f)} \) is nonzero. It then follows that there exists a prime ideal \( q \) in \( (S^{(d)})_{(f)} \). We will use this to construct a homogeneous prime \( p \) in \( S^{(d)} \) that does not contain \( f \) (and so in particular does not contain \( S_+^{(d)} \)), so hence the degree \( f \) is not nilpotent; the ideal generated by the homogeneous \( a \in S \) such that \( a^d \in S^{(d)} \) lies in \( p \) is then readily checked to be a homogeneous prime ideal of \( S \) that does not contain \( f \) (this rests crucially on the fact that membership in the homogeneous \( p \) may be checked on component-parts).

Let \( p \) be the contraction of \( q(S^{(d)})_f \) under \( S^{(d)} \to (S^{(d)})_f \). The ideal \( p \) of \( S^{(d)} \) does not contain \( f \), since otherwise \( q(S^{(d)})_f \) would contain the degree-0 element 1, which is absurd since \( q(S^{(d)})_f \subseteq (S^{(d)})_f = q \) is a proper ideal. To check that \( p \) is homogeneous prime, first observe that (by construction) \( q(S^{(d)})_f \) is an ideal of the \( \mathbb{Z} \)-graded \( (S^{(d)})_f \), so \( p \) is a homogeneous ideal of the \( \mathbb{N} \)-graded \( S^{(d)} \). Hence, to verify primality it is sufficient to work with homogeneous elements. That is, we consider homogeneous \( a, a' \in S^{(d)} \) with respective degrees \( dn \) and \( dn' \) and we assume \( a a' \in p \). Our goal is to prove \( a \in p \) or \( a' \in p \).

Since \( a a' \in p \), the homogenous image of \( a a' \) in \( S^{(d)}_f \) is contained in \( q(S^{(d)})_f \), so \( a a' = (x/f^n)f' \) with \( r \in \mathbb{Z} \), \( x \in S_{kd} \), and \( x/f^n \in q \subseteq (S^{(d)})_{(f)} \). Thus, by comparing degrees we get \( dn + dn' = dr \), so \( n + n' = r \). Hence, \( a a'/f'' = (a/f^n)(a'/f') \in (S^{(d)})_f \) is a product of terms with degree 0. However,

\[
\frac{a}{f^n} \frac{a'}{f'} \in (S^{(d)})_{(f)} \cap (q(S^{(d)})_f) = q,
\]

so by primality of \( q \) in \( (S^{(d)})_{(f)} \) we conclude that at least of the degree-0 elements \( a/f^n \) or \( a'/f' \) lies in \( q \). Hence, either \( a \) or \( a' \) in \( S^{(d)} \) map into \( q(S^{(d)})_f \) upon inverting \( f \), so by definition either \( a \) or \( a' \) lie in \( p \).
2. First steps towards a scheme structure

For homogeneous \( f \in S_+ \), we get an open set \( D_+(f) \subseteq \text{Proj}(S) \) consisting of those \( p \in \text{Proj}(S) \) that do not contain \( f \). These are a base of open sets, and we claim that \( D_+(f) \) is naturally homeomorphic to \( \text{Spec}(S_f) \), where \( S_f \subseteq S_f \) is the degree-0 part of the \( \mathbb{Z} \)-graded localization of \( S \) at the homogeneous \( f \).

To define a homeomorphism
\[
\varphi : D_+(f) \to \text{Spec}(S_f),
\]
to each \( p \in D_+(f) \) we associate the prime ideal
\[
p_{(f)} = (pS_f) \cap S_f \subseteq \text{Spec}(S_f);
\]
this is prime because it is the contraction of the prime \( p_S_f = p_f \) of \( S_f \) under the ring map \( S_f \to S_f \) (note that \( p_f \) is prime since \( p \) is a prime of \( S \) not containing \( f \)).

**Theorem 2.1.** The map \( \varphi : D_+(f) \to \text{Spec}(S_f) \) is a homeomorphism.

Proof. For any homogeneous ideal \( a \) of \( S \), we generalize the above operation on homogeneous prime ideals by defining
\[
\varphi(a) = (aS_f) \cap S_f.
\]
For any \( p \in D_+(f) \), we claim
\[
(1) \quad \varphi(a) \subseteq \varphi(p) \iff a \subseteq p.
\]
Once this is proved, it will follow that \( \varphi \) is at least injective. The \((\Leftarrow)\) implication is obvious, and for the converse it suffices to prove that if \( a \in a \) is a homogeneous element then \( a \in p \).

Let \( n = \deg a \geq 0 \) and let \( d = \deg f > 0 \). It follows that
\[
\frac{a^d}{f^n} \in aS_f \cap S_f = \varphi(a) \subseteq \varphi(p) = pS_f \cap S_f,
\]
so there exists a homogeneous \( x \in p \) such that \( a^d/f^n = x/f^m \) in \( S_f \) with \( md = \deg(x) \). Thus, for some \( e \geq 0 \) we have
\[
f^e(f^m a^d - f^n x) = 0
\]
in \( S \), and since \( f \not\in p \) we must have \( f^m a^d - f^n x \in p \). However, \( x \in p \), so \( f^m a^d \in p \). Since \( p \) is prime, \( f \not\in p \), and \( d \) is positive, we conclude \( a \in p \) as desired. This completes the proof of injectivity for \( \varphi \).

Once we prove \( \varphi \) is surjective, and hence is bijective, \((1)\) implies
\[
\varphi(V(a) \cap D_+(f)) = V(\varphi(a)).
\]
Hence, for any ideal \( b \) of \( S_f \), the preimage \( a \) of \( bS_f \) in \( S \) is a homogeneous ideal satisfying \( \varphi(a) = b \). We may therefore conclude that every closed set \( V(b) \) in \( \text{Spec}(S_f) \) corresponds (under the bijection \( \varphi \)) to a closed set \( V(a) \cap D_+(f) \) in \( D_+(f) \). However, all closed sets in \( D_+(f) \) (with the subspace topology from \( \text{Proj}(S) \)) have such a form for some \( a \), so we thereby get that \( \varphi \) is a homeomorphism.

It remains to check that \( \varphi \) is surjective. A key observation is that the natural map
\[
(S^{(d)}_{(f)}) \to S_f
\]
is an isomorphism. The basic idea is that a degree-0 element in \( S_f \) must have the form \( x/f^n \) with homogeneous \( x \) of degree \( \deg(x) = nd \in S_{nd} \), so \( x \) is in \( S^{(d)} \); the straightforward details are left to the reader (hint: equality of subrings of \( S_f \)). Via this identification, any prime ideal of \( S_f \) may be considered as a prime ideal in \( (S^{(d)}_{(f)}) \). However, in the proof of Theorem 1.3 it was proved (check!) that every prime ideal \( q \) of \( (S^{(d)}_{(f)}) \) has the form \( \varphi(p) \) for some homogeneous prime \( p \) of \( S \) not containing \( f \) (that is, for some \( p \in D_+(f) \)).

Let us now write \( \varphi_f : D_+(f) \to \text{Spec}(S_f) \) to denote the homeomorphism constructed above, with \( f \in S_+ \) any positive-degree homogeneous element (so \( \varphi_f(p) = pS_f \cap S_f \)). We shall use this homeomorphism to endow \( D_+(f) \) with a structure of affine scheme, using the structure sheaf on \( \text{Spec}(S_f) \). In view of the fact that the \( D_+(f) \)'s form a base of opens in \( \text{Proj}(S) \), the key issue is to identify \( S_f \) as the ring of sections
on the open subset \( D_+(f) \subset \text{Proj}(S) \), and to this end it is useful to note that \( S_{(f)} \) may be described entirely in terms of the subset \( D_+(f) \subset \text{Proj}(S) \) and the ring \( S \) without mentioning \( f \):

**Theorem 2.2.** For homogeneous \( f \in S_+ \), let \( T_f \) be the multiplicative set of homogeneous elements \( g \in S \) such that \( g \not\in p \) for all \( p \in D_+(f) \subset \text{Proj}(S) \) (despite the notation, \( T_f \) only depends on \( D_+(f) \) and not on \( f \)). The natural map

\[
S_{(f)} \rightarrow (T_f^{-1}S)_0
\]

to the degree-0 part of the \( \mathbb{Z} \)-graded \( T_f^{-1}S \) induced by \( S_f \rightarrow T_f^{-1}S \) is an isomorphism.

**Proof.** Let \( d = \deg f > 0 \). For injectivity, suppose \( x \in S \) is homogeneous of degree \( nd \) and the degree-0 element \( x/f^n \in S_f \) maps to 0 in \( T_f^{-1}S \). Hence, there exists \( g \in T_f \) such that \( gx = 0 \) in \( S \). Replacing \( g \) with \( g^d \) if necessary, we can assume \( \deg g = md \). Thus,

\[
(g/f^m)(x/f^n) = 0
\]

in \( S_f \), and hence this equality holds in \( S_{(f)} \). By the definition of \( T_f \), for all \( p \in D_+(f) \) we have \( g \not\in p \), so \( g/f^m \) is not contained in the prime ideal \( \varphi_f(p) = (pS_f) \cap S_{(f)} \) of \( S_{(f)} \) (as \( f \not\in p \)). But \( \varphi_f \) is bijective onto \( \text{Spec}(S_{(f)}) \), so \( g/f^m \in S_{(f)} \) is not contained in any primes. It follows that \( g/f^m \in S_{(f)} \) is a unit, so the vanishing of \( (g/f^m)(x/f^n) \) in \( S_{(f)} \) forces \( x/f^n = 0 \) in \( S_f \). This gives exactly the desired injectivity.

Now choose \( g \in T_f \) and \( x \in S \) with \( \deg(x) = \deg(g) \), so \( x/g \in (T_f^{-1}S)_0 \) makes sense. We seek a homogeneous \( a \in S \) of some degree \( nd \) (for some \( n \geq 0 \)) such that \( a/f^n \in S_{(f)} \) maps to \( x/g \) in \( T_f^{-1}S \). We may replace \( x \) with \( g^d^{-1}x \) and \( g \) with \( g^d \) to get to the case \( \deg g = md \) for some \( m \geq 0 \). Thus, using the definition of \( T_f \) and the bijectivity of \( \varphi_f \) we see that \( g/f^m \in S_{(f)} \) is not contained in any prime ideals, so it is a unit. In the degree-0 part of the \( \mathbb{Z} \)-graded \( T_f^{-1}S \) we have

\[
\frac{x}{g} = \frac{f^m}{g} \cdot \frac{x}{f^m}, \quad \frac{g}{f^m} \cdot \frac{x}{f^m} = \frac{f^m}{g} \cdot \frac{x}{f^m},
\]

so \( (g/f^m)^{-1}(x/f^n) \in S_{(f)} \) maps to \( x/g \in (T_f^{-1}S)_0 \). This proves the desired surjectivity.

3. A scheme structure on \( \text{Proj}(S) \)

By Theorem 2.2, whenever \( f, h \in S_+ \) are homogeneous elements such that \( D_+(h) \subset D_+(f) \) inside of \( \text{Proj}(S) \) we have (by the definitions!) \( T_f \subset T_h \) inside \( S \), and so we get a canonical map

\[
(2) \quad S_{(f)} = (T_f^{-1}S)_0 \rightarrow (T_h^{-1}S)_0 = S_{(h)}
\]

on degree-0 parts induced by the map \( T_f^{-1}S \rightarrow T_h^{-1}S \) of \( \mathbb{Z} \)-graded localizations. We may therefore consider the diagram of topological spaces

\[
\begin{array}{ccc}
D_+(f) & \xrightarrow{\varphi_f} & \text{Spec}(T_f^{-1}S)_0 \\
\downarrow \cong & & \downarrow \cong \\
D_+(h) & \xrightarrow{\varphi_h} & \text{Spec}(T_h^{-1}S)_0
\end{array}
\]

where the left column is the inclusion within \( \text{Proj}(S) \). One readily checks (upon reviewing the definitions of the various maps) that this diagram commutes with the right side an open embedding, ultimately because the canonical equality

\[
(S_{(f)})_{h^{\deg f}/f^{\deg h}} = S_{(h)} = (S_{(h)})_{f^{\deg h}/h^{\deg f}}
\]

inside of \( S_{fh} \) (check!) and the fact that \( f^{\deg h}/h^{\deg f} \in S_{(h)}^+ \) (since \( f \in T_f \subset T_h \)) implies that (2) induces an isomorphism \( (S_{(f)})_{h^{\deg f}/f^{\deg h}} \cong S_{(h)} \).

Clearly \( D_+(f) \cap D_+(g) = D_+(fg) \), and by taking \( h = fg \) above we see that this open subset of \( D_+(f) \) is carried by \( \varphi_f \) onto the open subset

\[
\text{Spec}((S_{(f)})_{g^{\deg f}/f^{\deg g}}) \subset \text{Spec}(S_{(f)}).
\]
Likewise, as an open subset of $D_+(g)$ it is carried by $\varphi_g$ onto the open subset
$$\text{Spec}((S(g))_{f^{\deg f}/g^{\deg g}}) \subseteq \text{Spec}(S(g)).$$

We now have put three scheme structures on $D_+(f)$, namely $\text{Spec} S(fg)$ and the two as basic opens in $\text{Spec} S(f)$ and in $\text{Spec} S(g)$. These three structures are identified by means of the ring isomorphisms
$$(3) \quad (S(f))_{g^{\deg f}/f^{\deg g}} \simeq S(fg) \simeq (S(g))_{f^{\deg f}/g^{\deg g}}$$
that are really equalities as subrings of $S_{fg}$. Consequently, the cocycle condition for gluing is satisfied (it comes down to transitivity for equality among three subrings of $S(fgh)$ for any three homogeneous $f, g, h \in S_+$, so we may glue the structure sheaves $O_{\text{Spec}(S(f))}$ over the $D_+(f)$’s via (3). That is, we are gluing the $\text{Spec} S(f)$’s (as ringed spaces) along the $\text{Spec} S(fg)$’s, where the underlying topological space $\text{Proj}(S)$ of the gluing was made at the start.

The glued structure sheaf over $P = \text{Proj}(S)$ will be denoted $O_P$, and so the ringed space $(P, O_P)$ is covered by open subspaces
$$(D_+(f), O_P|_{D_+(f)}) \simeq \text{Spec}(S(f))$$
for homogeneous $f \in S_+$. Hence, $(P, O_P)$ is a scheme.

**Definition 3.1.** Let $S$ be an $\mathbb{N}$-graded ring. The scheme $\text{Proj}(S)$ is the topological space denoted $\text{Proj}(S)$ above, equipped with the unique sheaf of rings $O_P$ whose restriction to $D_+(f)$ is $O_{\text{Spec}(S(f))}$ (using $\varphi_f$) for all homogeneous $f \in S_+$, with the overlap-gluing isomorphism
$$O_{\text{Spec}(S(f))_{f^{\deg f}/f^{\deg g}}} = O_{\text{Spec}(S(f))_{D_+(f) \cap D_+(g)}} \simeq O_{\text{Spec}(S(g))_{D_+(g) \cap D_+(f)}} \simeq O_{\text{Spec}(S(fg))_{f^{\deg f}/g^{\deg g}}}$$
defined by the isomorphism $\text{Spec}((S(f))_{g^{\deg f}/f^{\deg g}}) \simeq \text{Spec}((S(g))_{f^{\deg f}/g^{\deg g}})$ arising from the canonical ring isomorphism in (3) for homogeneous $f, g \in S_+$.

By Theorem 1.3, we obtain a useful alternative description:

**Corollary 3.2.** Let $\{f_i\}$ be a collection of homogeneous elements in $S_+$ such that every element of $S_+$ has some power contained in the ideal generated by the $f_i$’s. The scheme $\text{Proj}(S)$ is obtained by gluing the affine schemes $\text{Spec}(S(f_i))$ along the open affine overlaps $\text{Spec}(S(f_i, f_j)) \hookrightarrow \text{Spec}(S(f_i))$ defined by the isomorphisms
$$S(f_i, f_j) \simeq (S(f_i))_{f_j^{\deg f_i}/f_i^{\deg f_j}}.$$
Remark 3.5. If we assign \( A[X_0, \ldots, X_n] \) an \( \mathbb{N} \)-graded structure by putting \( A \) in degree 0 and assigning \( X_i \) some positive degree \( d_i \), the resulting \( \mathbb{N} \)-graded rings are generally \textit{not} isomorphic as \( \mathbb{N} \)-graded rings for different \( n \)-tuples \( \mathbf{d} = (d_0, \ldots, d_n) \), and their \( A \)-scheme Proj’s (so-called \textit{weighted projective} \( n \)-\textit{spaces over} \( A \) with weights \( \mathbf{d} \)) are generally \textit{not} isomorphic to each other.