Math 216A. Further properties of $\widetilde{M}$ for graded M

1. Setup and Hom

Let S be an $\mathbf{N}$-graded ring, and M and N two $\mathbf{Z}$-graded S-modules. (At the outset we don't assume S is generated by S_1 over S_0, but we will eventually impose this.) Let $X = \mathrm{Proj}(S)$.

For homogeneous $f \in S$ we discussed in class that there is a natural map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ that is an isomorphism when $f \in S_1$. In particular, there is always a natural map

$$\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to (M \otimes_S N)^{\sim} \tag{1}$$

that is an isomorphism over $D_+(f)$ for $f \in S_1$. Hence, as we discussed in class, (1) is an isomorphism when S is generated by S_1 over S_0.

On affine schemes, Hom and sheaf-Hom have been related through the functor from modules to quasi-coherent sheaves, and pullback and pushfoward under affine maps have been related to specific operations on modules. Our first aim is to establish analogues of those results in the graded setting. A new wrinkle is that (unlike tensor products) there isn't generally a grading on $\mathrm{Hom}_S(M, N)$ and (unlike for Spec) Proj isn't generally a functor for graded ring homomorphisms. We first focus on the situation with Hom, since it doesn't get involved with functoriality of Proj.

Definition 1.1. For $k \in \mathbf{Z}$ define $\mathrm{Hom}_S^k(M, N)$ to be the set of graded S-linear maps $M \to N(k)$ (i.e., S-linear maps $u : M \to N$ such that $u(M_r) \subset N_{r+k}$ for all $r \in \mathbf{Z}$, sometimes called "maps of degree k"). Define

$$\mathrm{Hom}_S^*(M, N) := \bigoplus_{k \in \mathbf{Z}} \mathrm{Hom}_S^k(M, N).$$

In [EGA II, 2.1.2] the construction $\mathrm{Hom}_S^*(M, N)$ is denoted as $\mathrm{Hom}_S(M, N)$, but we avoid that since it does not generally agree with the usual meaning of the latter notation. However, we will soon see that $\mathrm{Hom}_S^*(M, N)$ is always naturally an S-submodule of $\mathrm{Hom}_S(M, N)$ and that the two coincide under some finiteness hypotheses on M. Note that Definition 1.1 even makes sense for $\mathbf{Z}$-graded S (not just $\mathbf{N}$-graded S), such as the ring T_f for homogeneous f in an $\mathbf{N}$-graded T; this will be useful later.

If $f \in S_d$ and $u \in \mathrm{Hom}_S^k(M, N)$ then $fu : M \to N$ shifts degrees by $k + d$ (i.e., $fu \in \mathrm{Hom}_S^{k+d}(M, N)$), so the $\mathbf{Z}$-graded $\mathrm{Hom}_S^*(M, N)$ is naturally a graded S-module. There is an evident S-linear map

$$\mathrm{Hom}_S^*(M, N) \to \mathrm{Hom}_S(M, N) \tag{2}$$

by sending $(u_k)_k$ to $\sum_k u_k : m \mapsto \sum_k u_k(m)$ (which makes sense since $u_k = 0$ for all but finitely many k).

Lemma 1.2. *The map* (2) *is injective.*

Proof. Pick a finite collection of elements $u_j \in \mathrm{Hom}_S^{k_j}(M, N)$ for pairwise distinct k_j's. The S-linear map $u = \sum u_j : M \to N$ satisfies $u(m_d) = \sum_j u_j(m_d)$ for $m_d \in M_d$, with elements

$u_j(m_d) \in N$ that are homogeneous with pairwise distinct degrees $d + k_j$. Hence, the only way their sum can vanish in N is when $u_j(m_d) = 0$ for all j. Thus, if $u = 0$ then all u_j vanish on every M_d, so the desired injectivity holds. ■

Lemma 1.3. *If M is finitely presented as an S-module (i.e. it is the cokernel of a map between finite free S-modules – this has nothing to do with the grading) then the inclusion $\mathrm{Hom}_S^*(M,N) \hookrightarrow \mathrm{Hom}_S(M,N)$ is an equality. In other words, every S-linear $M \to N$ is a uniquely a sum of finitely many graded maps $M \to N(k_j)$.*

Remark 1.4. The proof of this lemma only needs that S is $\mathbf{Z}$-graded rather than $\mathbf{N}$-graded.

Proof. By definition, a finitely presented module over ring admits *some* surjection from a finite free module for which the kernel is finitely generated. But in fact then *every* surjection from a finite free module has a finitely generated kernel [Mat, Thm. 2.6]. Thus, upon choosing a graded surjection (as we may) $S(d_1) \oplus \cdots \oplus S(d_m) \twoheadrightarrow M$, the $\mathbf{Z}$-graded kernel is also finitely generated and hence is a quotient of another such direct sum. In other words, a finitely presented graded S-module is also finitely presented in a graded sense: it is a cokernel of a graded linear map between two finite direct sums of $S(d)$'s.

Both functors $\mathrm{Hom}_S^*(\cdot, N)$ and $\mathrm{Hom}_S(\cdot, N)$ carry right exact sequences of graded maps of graded S-modules to left-exact sequences. Thus, upon choosing a "finite graded presentation" (as we just saw can always be found)

$$\bigoplus_j S(n_j) \to \bigoplus_i S(d_i) \to M \to 0$$

we can apply both functors to arrive at a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_S^*(M,N) & \longrightarrow & \mathrm{Hom}_S^*(\oplus S(d_i), N) & \longrightarrow & \mathrm{Hom}_S^*(\oplus S(n_j), N) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_S(M,N) & \longrightarrow & \mathrm{Hom}_S(\oplus S(d_i), N) & \longrightarrow & \mathrm{Hom}_S(\oplus S(n_j), N) \end{array}$$

with all vertical maps injective (Lemma 1.2). Hence, to show the first vertical arrow is an isomorphism it suffices to show the same for the other two. That is, we are reduced to the case M is a finite direct sum of $S(d)$'s.

Everything in sight behaves well for direct sums, so we can assume $M = S(d)$ for some $d \in \mathbf{Z}$. We have $\mathrm{Hom}_S^*(S(d), N) = N(-d)$ (chase graded parts) and $\mathrm{Hom}_S(S(d), N) = \mathrm{Hom}_S(S, N) = N$, and this identifies the inclusion $\mathrm{Hom}_S^*(S(d), N) \to \mathrm{Hom}_S(S(d), N)$ with the inclusion $j : N(-d) \hookrightarrow N$ that carries $N(-d)_r = N_{r-d}$ into N via the natural inclusion. Summing these over all $r \in \mathbf{Z}$ yields j as the natural identity map on underlying abelian groups. ■

To express how Hom and sheaf-Hom are related in the graded setting, we shall first build an $\mathscr{O}_X$-linear map

$$\theta : \mathrm{Hom}_S^*(M,N)^{\sim} \to \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \tag{3}$$

without hypotheses on S or M, and then will see that it is an isomorphism under some mild hypotheses. To define θ, we will build a map between sections over $D_+(f)$ for all homogeneous $f \in S_+$ (i.e., $f \in S_d$ for some $d > 0$) and then consider compatibility under change in f (to ensure the constructions glue to a map between sheaves on X). **Warning**: remember that sheaf-Hom applied to quasi-coherent sheaves is generally *not* quasi-coherent.

Building θ on sections over $D_+(f)$ amounts to making an $S_{(f)}$-linear map

$$\mathrm{Hom}_S^*(M,N)_{(f)} \to \mathrm{Hom}_{D_+(f)}(\widetilde{M}|_{D_+(f)}, \widetilde{N}|_{D_+(f)}) = \mathrm{Hom}_{D_+(f)}(\widetilde{M_{(f)}}, \widetilde{N_{(f)}}).$$

But $\widetilde{(\cdot)}$ is a functor from modules to quasi-coherent sheaves on any affine scheme, so the right side here receives a natural map from $\mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$. Thus, to define θ on sections over $D_+(f)$ it suffices to build an $S_{(f)}$-linear map $\mathrm{Hom}_S^*(M,N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$. Naturally one has

$$\mathrm{Hom}_S^*(M,N)_{(f)} \subset \mathrm{Hom}_S^*(M,N)_f \subset \mathrm{Hom}_S(M,N)_f \to \mathrm{Hom}_{S_f}(M_f, N_f)$$

via the functoriality of localization for the final step. A direct calculation with the definition of $\mathrm{Hom}_S^*(M,N)$ shows that this composite map lands inside the set of S_f-linear maps $M_f \to N_f$ that respect the $\mathbf{Z}$-gradings on M_f and N_f, so in particular carries $M_{(f)}$ into $N_{(f)}$ (necessarily $S_{(f)}$-linearly). In this way we define θ on sections over $D_+(f)$. From the construction one computes that it behaves well with respect to restriction along an inclusion of open sets $D_+(fg) \subset D_+(f)$, so it defines the sheaf map θ.

Proposition 1.5. *If M is a finitely presented S-module (so $\mathrm{Hom}_S^*(M,N) = \mathrm{Hom}_S(M,N)$ by Lemma 1.3) and S is generated by S_1 over S_0 then θ in (3) is an isomorphism.*

Proof. Let's check that under the given hypotheses on S and M, the source and target of θ are *quasi-coherent*. For the source it is by design, and for the target it is because (by reducing to the affine case) for any scheme Y the sheaf $\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G})$ is quasi-coherent for $\mathscr{G}$ that is quasi-coherent and $\mathscr{F}$ that is "finitely presented": locally a cokernel of a map between finite free sheaves (e.g., $\widetilde{M}$ on X is finitely presented, as seen by expressing M as a cokernel of a map between $S(d)$'s and using the local freeness of each $\widetilde{S(d)}$ due to S being generated by S_1 over S_0!).

Thus, to check the isomorphism property it suffices to check bijectivity between sets of sections over the members of a single open cover of X, such as on the $D_+(f)$'s for elements $f \in S_1$. For much of what follows we will use only that f is homogeneous; that it is in degree 1 will be used at the end of the proof.

The map on $D_+(f)$-sections has the form

$$\mathrm{Hom}_S(M,N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$$

(for the target, we are using that the functor $\widetilde{(\cdot)}$ from modules to quasi-coherent sheaves on any affine scheme, such as $\mathrm{Spec}(S_{(f)})$, is fully faithful); we want to show this is an isomorphism. The source is the degree-0 part of $\mathrm{Hom}_S(M,N)_f$ (which has a $\mathbf{Z}$-grading by Remark 1.4), and the target receives a map from $\mathrm{Hom}^0_{S_f}(M_f, N_f)$ (the set of $\mathbf{Z}$-graded maps over the $\mathbf{Z}$-graded ring S_f). Since M is a finitely presented S-module, the functor $\mathrm{Hom}_S(M, \cdot)$ on S-modules (no grading) commutes with any localization, so naturally

$$\mathrm{Hom}_S(M,N)_f \simeq \mathrm{Hom}_{S_f}(M_f, N_f). \tag{4}$$

The source and target in (4) are naturally $\mathbf{Z}$-graded (using Remark 1.4 for the target), and the map (4) respects the gradings, so it is an isomorphism of graded modules (as any graded map that is an isomorphism on underlying abelian groups has a graded inverse). In this way $\mathrm{Hom}_S(M,N)_{(f)}$ is *identified with* (not just receives a map from!) $\mathrm{Hom}^0_{S_f}(M_f,N_f)$.

The effect of θ on $D_+(f)$-sections is now identified with the natural map

$$\theta^0_f : \mathrm{Hom}^0_{S_f}(M_f,N_f) \to \mathrm{Hom}_{S_{(f)}}(M_{(f)},N_{(f)})$$

that assigns to a graded homomorphism its effect on degree-0 parts, so our task is reduced to showing that θ^0_f is an isomorphism for any finitely presented graded S-module M and any $f \in S_1$. Both source and target of θ^0_f carry a right-exact sequence in M's (with graded maps) to a left-exact sequence. Thus, by the same argument as in the proof of Lemma 1.3 (using that M is finitely presented) we reduce to the case $M = S(d)$ for some $d \in \mathbf{Z}$.

With $M = S(d)$, the source of θ^0_f is identified with the part of N_f in degree $-d$ (since $S(d)_f = S_f(d)$ and for a $\mathbf{Z}$-graded module P over a $\mathbf{Z}$-graded ring T – such as the S_f-module N_f – the graded maps $T(d) \to P$ are exactly $t \mapsto tp$ for unique $p \in P_{-d}$). The target of θ^0_f is $\mathrm{Hom}_{S_{(f)}}(S(d)_{(f)},N_{(f)})$ and $S(d)_{(f)}$ is identified with the part of S_f in degree d, so θ^0_f is identified with a map

$$\alpha_{f,d} : (N_f)_{-d} \to \mathrm{Hom}_{S_{(f)}}((S_f)_d,N_{(f)})$$

that is readily checked to be the one arising from $(N_f)_{-d} \otimes_{S_{(f)}} (S_f)_d \to N_f$ (via multiplication). Hence, it suffices to show that the maps $\alpha_{f,d}$ are isomorphisms.

Now we finally use crucially that f has degree 1: it ensures (check!) $(S_f)_d$ is a free $S_{(f)}$-module with basis f^d. Likewise, $(N_f)_{-d} \simeq N_{(f)}$ as $S_{(f)}$-modules via the $S_{(f)}$-linear automorphism of N_f defined by multiplication by f^d. In this way, $\alpha_{f,d}$ is identified with the map $N_{(f)} \to \mathrm{Hom}_{S_{(f)}}(S_{(f)},N_{(f)})$ given by $\nu \mapsto (h \mapsto h\nu)$ that is obviously an isomorphism. ■

2. Interaction with Proj "functoriality"

Having dealt with Hom and sheaf-Hom, we next relate $\widetilde{M}$ for graded M with pushforward and pullback. The main initial issue is that Proj is not functorial as Spec is. Consider a general map $\varphi : S \to T$ of $\mathbf{N}$-graded rings. Let $U \subset \mathrm{Proj}(T)$ be the open subscheme complementary to $V(\varphi(S_+)T)$. In our initial discussion of Proj in class, we built a map schemes $f : U \to \mathrm{Proj}(S)$ given on underlying sets by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ and defined at the level of ringed spaces gluing the maps

$$\mathrm{Spec}(T_{(\varphi(s))}) = D_+(\varphi(s)) \to D_+(s) = \mathrm{Spec}(S_{(s)})$$

for homogeneous $s \in S_+$ arising from $S_{(s)} \to T_{(\varphi(s))}$ defined by applying the graded map φ to degree-0 fractions.

The compatibility of $\widetilde{(\cdot)}$ on graded modules with the functors f_* and f^* on sheaves takes on the following form that is reminiscent of the affine case with Spec.

Proposition 2.1. *Let M be a graded S-module and N a graded T-module.*

(i) *There is a natural map $f^*(\widetilde{M}) \to (T \otimes_S M)^{\sim}|_U$, and it is an isomorphism when S is generated by S_1 over S_0 (no hypotheses on T).*

(ii) *For the associated "underlying" graded S-module ${}_SN$ via composition with φ, there is a natural isomorphism $\widetilde{{}_SN} \simeq f_*(\widetilde{N}|_U)$. (No hypotheses on S or T.)*

Proof. Note that f is an *affine* map: its target $\mathrm{Proj}(T)$ is covered by affine opens $D_+(s)$ for homogeneous $s \in S_+$, and $f^{-1}(D_+(s)) = D_+(\varphi(s))$ since f on underlying sets is given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. Hence, by our knowledge about pullback and pushforward for quasi-cohernt sheaves relative to affine morphisms, in both (i) and (ii) the desired maps will at least go between quasi-coherent sheaves.

The desired maps will be built by gluing over an affine open cover. For (ii), if $s \in S_+$ is homogeneous then since $f^{-1}(D_+(s)) = D_+(\varphi(s))$ we see that over $D_+(s)$ the isomorphism we want to make between quasi-coherent sheaves corresponds to an isomorphism of $S_{(s)}$-modules $({}_SN)_{(s)} \to {}_{S_{(s)}}(N_{(\varphi(s))})$: there is an evident such isomorphism arising from the effect of the natural graded (!) isomorphism of graded S_s-modules $({}_SN)_s \to {}_{S_s}(N_{\varphi(s)})$ when restricted to degree-0 parts, and these isomorphisms in degree 0 are easily compatible with inclusions $D_+(ss') \subset D_+(s)$ for homogeneous $s, s' \in S_+$. This settles (ii).

Now consider (i). By definition, U is covered by the affine open subschemes $D_+(\varphi(s))$ for homogeneous $s \in S_+$. Over such an open set, the desired map must arise from a $T_{\varphi(s)}$-linear map

$$T_{\varphi(s)} \otimes_{S_{(s)}} M_{(s)} \to (T \otimes_S M)_{(\varphi(s))}. \tag{5}$$

There is an evident such map defined via multiplication of degree-0 fractions, and these are easily seen to glue to a global map in the usual way (comparing with open immersions $D_+(\varphi(s)\varphi(s')) \subset D_+(\varphi(s))$ for homogeneous $s, s' \in S_+$). This defines the map in (i) in general.

It remains to show that the map in (i) is an isomorphism when S is generated by S_1 over S_0. In such cases $\mathrm{Proj}(S)$ is covered by the affine opens $D_+(s)$ for $s \in S_1$, so likewise U is covered by the affine open preimages $D_+(\varphi(s))$ for $s \in S_1$. Hence, by quasi-coherence it suffices to verify the isomorphism property between the modules of sections over such open subsets. In other words, we want to show that if $s \in S_1$ then the natural map of $T_{\varphi(s)}$-modules in (5) is bijective. For any graded S-module N there is a natural map $N_{(s)} \otimes_{S_{(s)}} M_{(s)} \to (N \otimes_S M)_{(s)}$ which we know is bijective for $s \in S_1$. Applying this to $N = {}_ST$ (viewing T as a graded S-module via φ) then does the job. ■

Example 2.2. Here is a useful application of part (ii) of the preceding result. Pick $d > 0$. The grading on $S^{(d)}$ has all nonzero parts in degree divisible by d, so we can adjust the grading by dividing it by d everywhere (so S_d is the "degree-1 part"); i.e., declare the degree-n part to be S_{nd}. This adjustment in the definition of the grading maps j no longer has *no effect* on Proj, since what constitutes a homogeneous element or a degree-0 fraction is unaffected. However, this shift in the grading makes $S^{(d)}(1)$ more interesting, since its degree-0 part is now S_d (whereas with the initial grading coming from S it would vanish when $d > 1$). In the Proj handout we built an isomorphism $f : X = \mathrm{Proj}(S) \simeq \mathrm{Proj}(S^{(d)}) = X_d$ using the grading on $S^{(d)}$ coming from S.

If we use the grading after division by d then f isn't coming from a map between the resulting graded rings, but $S^{(d)}(1)$ is now S_d whereas before it was 0 when $d > 1$. Use this "divided" notion of degree to define $\mathscr{O}_{X_d}(1)$. We claim that $f_*(\mathscr{O}_X(d)) \simeq \mathscr{O}_{X_d}(1)$ (so, since

f is an isomorphism, likewise $\mathscr{O}_X(d) \simeq f^*(\mathscr{O}_{X_d}(1)))$. At the end of this example, we explain the meaning of this isomorphism in the important case $X = \mathbf{P}^n_A$ for a rng A.

Since f comes from the map of graded rings $S^{(d)} \hookrightarrow S$ when we don't divide the degrees by d on $S^{(d)}$, $f_*(\mathscr{O}_X(d))$ corresponds to the graded $S^{(d)}$-module underlying the graded S-module $S(d)$ using the "non-divided" degree. We want to relate this to the graded $S^{(d)}$-module $S^{(d)}(1)$ in the "divided degree" setting, which is exactly $S^{(d)}(d)$ in the "non-divided" degree setting. Making such division on the degree yields the *same* quasi-coherent sheaf on X_d (since there is no effect on degree-0 fractions in module localizations at homogeneous elements of the graded ring), so our task is finally reduced to the following in the "non-divided" degree setting: show the inclusion $M = S^{(d)}(d) \hookrightarrow S(d) = M'$ of graded $S^{(d)}$-modules (when $S^{(d)}$ is given its d-divisible degree from S) induces an *isomorphism* on the associated quasi-coherent sheaves.

The functor $\widetilde{(\cdot)}$ from graded modules to quasi-coherent sheaves is exact, so the natural map $\widetilde{M} \to \widetilde{M'}$ is certainly injective. To show it is an equality, it suffices to analyze sections over the base of opens $D_+(f)$ for homogeneous $f \in S^{(d)}_+$, which is to say that we want the inclusion $M_f \to M'_f$ of $\mathbf{Z}$-graded $S^{(d)}$-modules to be an equality on degree-0 parts. But this is obvious: any degree-0 fraction m'/f^r for homogeneous $m' \in M'$ can be written with exponent r divisible by d, so then m' viewed as an element of $S(d)$ has degree divisible by d, so it comes from $S^{(d)}(d)$.

Let's now consider the special case $S = A[t_0, \dots, t_n]$. Here $S^{(d)}$ is the A-subalgebra of polynomials only involving monomials of degree divisible by d, so it is generated by S_d over A, and upon writing $S^{(d)}$ with its "divided" degree as a quotient of a polynomial ring over A with one variable per degree-d monomial in the t_i's, we get a closed immersion of $\iota_d : \mathbf{P}^n_A = X \simeq X_d \hookrightarrow \mathbf{P}^N_A$ (the "d-uple embedding") under which $\mathscr{O}_{\mathbf{P}^N_A}(1)$ pulls back to $\mathscr{O}_{X_d}(1)$ that we have seen is identified with $\mathscr{O}_X(d)$.

By the universal property of $\mathbf{P}^N_A$ from Exercise B in HW9, ι_d corresponds to the line bundle $\mathscr{O}_X(d)$ equipped with its collection of global sections arising from the elements of $S(d)_0 = S_d$ given by the degree-d monomials in S (these "generate" the line bundle stalkwise since even just the power monomials t_j^d do the job: $X = \mathbf{P}^n_A$ is covered by the open affines $D_+(t_j^d)$, on which $\mathscr{O}_X(d)$ corresponds to the $S_{(t_j)}$-module $S(d)_{(t_j)} = S_{(t_j)} t_j^d$). The map ι_d is given on "homogeneous coordinates" by $[t_i] \mapsto [M_\alpha(t)]$ with M_α varying through the degree-d monomials in the t's, and the geometric meaning of the isomorphism $\mathscr{O}_X(d) \simeq \iota_d^*(\mathscr{O}_{X_d}(1))$ when $A = k$ is a field is that any hyperplane in the target $\mathbf{P}^N_k$ meets the source $\mathbf{P}^n_k$ in a degree-d hypersurface, since if a linear form in the M_α's is expressed in terms of the t_i's then it becomes a homogeneous polynomial of dgeree d!

3. Relating $\Gamma_*(\cdot)$ and $\widetilde{(\cdot)}$

For $X = \mathrm{Proj}(S)$, we now discuss the relationship between the functor Γ_* from $\mathscr{O}_X$-modules to graded S-modules and the functor $\widetilde{(\cdot)}$ from graded S-modules to quasi-coherent $\mathscr{O}_X$-modules.

By definition in class, $\Gamma_*(\mathscr{F}) = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F}(n))$ with $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$; this was made into a graded S-module via the map of $\mathbf{N}$-graded rings $\alpha_S : S \to \Gamma_*(\mathscr{O}_X)$. In class, when S *is generated by* S_1 *over* S_0 we built two maps: for any graded S-module M we made a graded map $\alpha_M : M \to \Gamma_*(\widetilde{M})$, and for any $\mathscr{O}_X$-module $\mathscr{F}$ we made a map of $\mathscr{O}_X$-modules $\beta_{\mathscr{F}} : \Gamma_*(\mathscr{F})^\sim \to \mathscr{F}$. The link between these that we want to address here is:

Theorem 3.1. *Assume S is generated by S_1 over S_0. Then for any graded S-module M, the composite map of sheaves $\widetilde{M} \xrightarrow{\widetilde{\alpha_M}} \Gamma_*(\widetilde{M})^\sim \xrightarrow{\beta_{\widetilde{M}}} \widetilde{M}$ is the identity map.*

Proof. By quasi-coherence of $\widetilde{M}$, it suffices to check the effect is the identity map on sections over the open subschemes $D_+(f)$ for homogeneous $f \in S_1$, as these cover X due to the hypothesis on S. This is an $S_{(f)}$-linear map $M_{(f)} \to \Gamma_*(\widetilde{M})_{(f)} \to M_{(f)}$, so we just need to chase the effect on a degree-0 fraction m/f^r for homogenous $m \in M_r$. The first map is the effect on degree-0 fractions arising from f-localization of the graded map $\alpha_M : M \to \Gamma_*(\widetilde{M})$ between graded S-modules which in each degree $n \geq 0$ carries M_n into $\Gamma(X, \widetilde{M}(n))$ via the map

$$M_n = M(n)_0 \to \Gamma(X, \widetilde{M(n)}) \simeq \Gamma(X, \widetilde{M}(n))$$

(final equality using that S is generated by S_1 over S_0).

Thus, we want to show that $\beta_{\widetilde{M}}(\alpha_M(m)/f^r) = m/f^r$ on $D_+(f)$-sections. By definition of $\beta_{\mathscr{F}}$ for general $\mathscr{F}$, $\beta_{\widetilde{M}}(\alpha_M(m)/f^r)$ is defined using $S(-r)_{(f)} \to \Gamma(D_+(f), \mathscr{O}_X(-r))$ applied to $f^{-r} \in S(-r)_{(f)}$. More specifically, it is the image of $\alpha_M(m) \otimes f^{-r}$ under the composite map

$$\Gamma(D_+(f), \widetilde{M(r)}) \otimes \Gamma(D_+(f), \mathscr{O}_X(-r)) \to \Gamma(D_+(f), \widetilde{M(r)} \otimes \mathscr{O}_X(-r)) = \Gamma(D_+(f), \widetilde{M}),$$

the final step using again that S is generated by S_1 over S_0 in order to identify $\widetilde{M(n)}$ with $\widetilde{M}(n)$ and to identify $\mathscr{O}_X(-n)$ as dual to the invertible $\mathscr{O}_X(n)$ for all $n \geq 0$.

In other words, via the composite map

$$\widetilde{M(r)} \otimes \widetilde{S(-r)} \simeq \widetilde{M} \otimes \widetilde{S(r)} \otimes \widetilde{S(-r)} \to \widetilde{M} \otimes (S(r) \otimes_S S(-r))^\sim \simeq \widetilde{M} \tag{6}$$

we want to show that the effect on $D_+(f)$-sections carries $\alpha_M(m) \otimes f^{-r}$ to m/f^r. Since $f \in S_1$, the $S_{(f)}$-module $S(n)_{(f)}$ is free of rank 1 with basis f^n for all $n \in \mathbf{Z}$. The isomorphism $\widetilde{M} \otimes \widetilde{S(n)} \simeq \widetilde{M(n)}$ over $D_+(f)$ corresponds to the map

$$M_{(f)} \otimes_{S_{(f)}} (S_{(f)} f^n) \to M(n)_{(f)}$$

defined by $(\mu/f^q) \otimes (hf^n) \mapsto (h)(\mu f^{n-q})$.

Setting $\mu = m$, $q = r$, $n = r$, and $h = 1$, we see that $\alpha_M(m)|_{D_+(f)} \in M(r)_{(f)}$ corresponds to $(m/f^r) \otimes f^r$. Thus, on $D_+(f)$-sections the first step in (6) carries $\alpha_M(m) \otimes f^{-r}$ to $(m/f^r) \otimes f^r \otimes f^{-r}$, and the second step in (6) carries this to $(m/f^r) \otimes (f^r \otimes f^{-r})$ viewing f^n as an element of $S(n)_{(f)} = \Gamma(D_+(f), \widetilde{S(n)})$ for any $n \in \mathbf{Z}$. The multiplication isomorphism $S(r) \otimes_S S(-r) \to S$ of graded S-modules induces on degree-0 parts of f-localizations

$$S(r)_{(f)} \otimes_{S_{(f)}} S(-r)_{(f)} = (S(r) \otimes_S S(-r))_{(f)} \to S_{(f)}$$

(equality using that $f \in S_1$) which carries $f^r \otimes f^{-r}$ to 1. Hence, (6) carries $\alpha_M(m) \otimes f^{-r}$ to $(m/f^r) \cdot 1 = m/f^r \in M_{(f)} = \Gamma(D_+(f), \widetilde{M})$, as desired. ■