# Math 216A. Gluing ringed spaces

If $X$ is a ringed space and $\{X_i\}$ is a collection of open subspaces then it is a pleasant exercise (please do it) with the definitions to check that, as in the case of topological spaces, for any ringed space $Y$ and morphisms $f_i : X_i \to Y$ that agree on the open overlaps (i.e., $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ as morphisms $X_i \cap X_j \rightrightarrows Y$) there exists a unique morphism $f : X \to Y$ such that $f|_{X_i} = f_i$ as morphisms $X_i \rightrightarrows Y$ for all $i$.

Our aim in this handout is to run such a process in reverse: given ringed spaces $X_i$ we want to create a "gluing" $X$ of the $X_i$'s along suitable open "overlaps". We shall work with ringed spaces, but the same procedures work verbatim for locally ringed spaces, or (locally) ringed spaces of $k$-algebras for a commutative ring $k$, etc.

Here is the setup for abstract gluing. Let $\{X_i\}$ be a collection of ringed spaces, $X_{ij} \subset X_i$ a collection of open subspaces (i.e., open subsets equipped with the restricted sheaf of rings, meant to eventually play the role of "$X_i \cap X_j$" in the glued ringed spaces to be made), and $\varphi_{ij} : X_{ij} \simeq X_{ji}$ a collection of isomorphisms (of ringed spaces) such that:

(i) $X_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i$,
(ii) $\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$ for all $i, j, k$ (informally, this corresponds to the necessary condition $(X_i \cap X_j) \cap (X_i \cap X_k) = (X_j \cap X_k) \cap (X_j \cap X_i)$),
(iii) (cocycle or triple-overlap condition) $\varphi_{jk}|_{X_{jk} \cap X_{ji}} \circ \varphi_{ij}|_{X_{ij} \cap X_{ik}} = \varphi_{ik}|_{X_{ik} \cap X_{ij}}$ for all $i, j, k$.

Under these conditions, the key result is this:

**Theorem**. *There exists a ringed space $X$* **equipped with** *an open cover by open subspaces $U_i$ and isomorphisms $\varphi_i : X_i \simeq U_i$ (as ringed spaces) for all $i$ (i.e., open immersions $X_i \hookrightarrow X$ that cover the underlying topological space) such that the following universal mapping property holds: for* any *ringed space $Y$ there is a bijection*

$$\mathrm{Hom}(X, Y) \simeq \{(f_i) \in \prod_i \mathrm{Hom}(X_i, Y) \mid f_j|_{X_{ji}} \circ \varphi_{ij} = f_i|_{X_{ij}} \text{ for all } i, j\}$$

*via $f \mapsto (f|_{U_i} \circ \varphi_i)$.*

*Moreover, $\varphi_i^{-1}(U_i \cap U_j) = X_{ij}$ and the diagram*

$$\begin{array}{ccc} X_{ij} & \xrightarrow[\simeq]{\varphi_i} & U_i \cap U_j \\ {\scriptstyle \varphi_{ij}} \downarrow {\scriptstyle \simeq} & & \| \\ X_{ji} & \xrightarrow[\varphi_j]{\simeq} & U_j \cap U_i \end{array}$$

*commutes for all $i, j$.*

We call $(X, \{\varphi_i\})$ the *gluing of the $X_i$'s along the $\varphi_{ij}$'s*, and it is unique up to unique isomorphism (as whenever we characterize an object via a universal mapping property; please think through how that goes in this case).

Let's sketch the proof of this result, describing the main constructions and leaving the verification that they work as an instructive exercise in working through the various definitions. On underlying topological spaces, we define $|X|$ to be the quotient set

$$(\coprod |X_i|)/\sim$$

of the disjoint union of the $|X_i|$'s modulo the equivalence relation $x_i \sim x_j$ when $x_i \in X_{ij}$, $x_j \in X_{ji}$, and $x_j = \varphi_{ij}(x_i)$ (this is an equivalence relation due to the hypotheses on the given data, with transitivity relying on the cocycle condition!). This quotient set is given the quotient topology. Please check via the hypotheses on the given data that $|X_i| \to |X|$ is injective and in fact an open embedding of topological spaces, with $|X_i| \cap |X_j|$ inside $|X|$ equal to both $|X_{ij}|$ and $|X_{ji}|$ for all $i, j$ (again, using the cocycle condition!).

We next define the sheaf of rings $\mathscr{O}$ on $|X|$: for any open $U \subset |X|$, motivated by the sheaf axioms and the fact that $U$ is covered by the open subsets $U \cap |X_i|$ whose overlaps are $U \cap |X_{ij}| = U \cap |X_{ji}|$ inside $|X|$, we define

$$\mathscr{O}(U) = \{(s_i) \in \prod_i \mathscr{O}_{X_i}(U \cap |X_i|) \mid s_i|_{U \cap |X_{ij}|} = s_j|_{U \cap |X_{ji}|} \text{ via } \varphi_{ij}^{\#} \text{ for all } i, j\}.$$

You should check that this is a sheaf on $|X|$, that in a natural way the restriction $\mathscr{O}|_{|X_i|}$ is identified with $\mathscr{O}_{X_i}$ (so each $X_i = (|X_i|, \mathscr{O}_{X_i})$ is thereby an open subspace of $(|X|, \mathscr{O})$), and that the ringed space $X := (|X|, \mathscr{O})$ *equipped with* the identifications of the $X_i$'s as open subspaces satisfies all of the asserted properties in the Theorem.