# Math 216A. Finiteness aspects of graded modules and algebras

## 1. Main results

Let $S$ be an $\mathbf{N}$-graded ring finitely generated over $S_0$. Each generator is a finite sum of homogeneous elements, so we get a finite generating set consisting of homogeneous elements which we may assume are in positive degree (since anything in degree 0 belong to $S_0$ and hence is redundant for the purposes of a generating set over $S_0$). In other words, there is a graded surjection $S_0[X_0, \dots, X_m] \twoheadrightarrow S$ where $X_j$ is assigned some degree $\delta_j > 0$. Thus, we get a closed immersion

$$\operatorname{Proj}(S) \hookrightarrow \operatorname{Proj}(S_0[X_0, \dots, X_m])$$

into a "weighted projective space" over $S_0$. But is $\operatorname{Proj}(S)$ actually projective over $S_0$ (i.e, a closed subscheme of some $\operatorname{Proj}(S_0[t_0, \dots, t_q])$ where each $t_j$ is homogeneous of degree 1)? One purpose of this handout is to give an affirmative answer.

It will be convenent to express most of our work in terms of graded modules, so let $M$ be a $\mathbf{Z}$-graded $S$-module that is finitely generated as an $S$-module. Any element of $M$ is a finite sum of homogeneous elements, so $M$ has a finite generating set over $S$ consisting of homogeneous elements $m_1, \dots, m_r$. If $d_j = \deg(m_j)$ (i.e., $m_j \in S_{d_j}$) then we have a graded $S$-linear maps $S(-d_j) \to M$ via $s \mapsto sm_j$, so a graded $S$-linear surjection

$$\pi : \bigoplus_{j=1}^{r} S(-d_j) \twoheadrightarrow M.$$

Recall that by definition

$$S^{(d)} = \bigoplus_{q \geq 0} S_{dq}, \quad M^{(d)} = \bigoplus_{q \in \mathbf{Z}} M_{dq}.$$

In §2 we will prove:

**Proposition 1.1.** *Let $S$ and $M$ be as above.*

(i) *For sufficiently negative $n$ we have $M_n = 0$, and for all $n$ the $S_0$-module $M_n$ is finitely generated (e.g., $S_n$ is $S_0$-finite for all $n$).*

(ii) *There exists $d_0 > 0$ depending only on $S$ so that for all $n$ sufficiently large depending on $M$, we have $S_{d_0} M_n = M_{n+d_0}$.*

(iii) *For all $d > 0$ and $0 \leq r \leq d-1$, the $S^{(d)}$-module $\oplus_{q \in \mathbf{Z}} M_{dq+r}$ is finitely generated. (Using $r = 0$, this implies $M^{(d)}$ is $S^{(d)}$-finite for all $d > 0$.)*

(iv) *For all $d > 0$, the $\mathbf{N}$-graded $S_0$-algebra $S^{(d)}$ is finitely generated. For some large $d$, $S^{(d)}$ is generated by $S_d$ over $S_0$ (so likewise for all multiples of $d$).*

The importance of (iv) is that for $d$ big enough and divisible enough we have that the $S_0$-algebra $S^{(d)} = \oplus_{q \geq 0} S_{dq}$ is generated by the $S_0$-finite $S_d$, so if we redefine the $d\mathbf{Z}$-valued degree on $S^{(d)}$ by dividing it by $d$ (as we may) then $S^{(d)}$ is finitely generated in degree 1 over its degree-0 part. Hence, for such $d$ we have a closed immersion $\operatorname{Proj}(S) \simeq \operatorname{Proj}(S^{(d)}) \hookrightarrow \mathbf{P}^N_{S_0}$ over $S_0$. Thus, $\operatorname{Proj}(S)$ is projective over $S_0$!

Here is an important application of parts (ii) and (iii).

**Corollary 1.2.** *If $S$ is finitely generated by $S_1$ over $S_0$ and $M$ is $S$-finite then the quasi-coherent $\widetilde{M}$ is of finite type (i.e., locally corresponds to a finitely generated module, so for noetherian $S$ it is coherent), and $\widetilde{M} = 0$ if and only if $M_n = 0$ for all large $n$.*

*Proof.* Let $f_1, \dots, f_r \in S_1$ be a finite set of elements generating $S$ over $S_0$, so $\mathrm{Proj}(S)$ is covered by the open affines $D_+(f_j)$. If $f \in S_+$ is homogeneous of degree $d > 0$ then $S_{(f)} \simeq S^{(d)}/(f-1)$ and compatibly $M_{(f)} \simeq M^{(d)}/(f-1)$. Since $M^{(d)}$ is $S^{(d)}$-finite by (iii), we get the "finite type" assertion by letting $f$ vary through the $f_j$'s.

Now consider the vanishing assertion. We have $\widetilde{M} = 0$ if and only if each the quasi-coherent $\widetilde{M}|_{D_+(f_j)} = \widetilde{M_{(f_j)}}$ on $\mathrm{Spec}(S_{(f_j)})$ vanishes for each $j$, which is to say (by quasi-coherence!) $M_{(f_j)} = 0$ for each $j$. Every degree-0 fraction in $M_{(f_j)}$ can be written with denominator as big a power of $f_j$ as we please, so its homogeneous numerator has degree as large as we want. Thus, if $M_n = 0$ for all big $n$ then every $M_{(f_j)}$ vanishes and so $\widetilde{M} = 0$. (This implication did not use that $M$ is $S$-finite.)

For the converse, suppose $\widetilde{M} = 0$. We can pick a finite set of homogeneous generators $m_1, \dots, m_n$ of $M$, so if $d_j = \deg(m_j)$ (which might be negative!) then $m_j/f_j^{d_j}$ is a degree-0 fraction, so it vanishes in $M_{(f_j)} \subset M_{f_j}$. Hence, for some big $N_j$ we have $f_j^{N_j} m_j = 0$ in $M$. For $N = \max_j N_j$ we have $f_j^N m_j = 0$ for all $j$. Since the $f_j$'s generate $S$ over $S_0$, anything in $S$ with sufficiently big degree is an $S_0$-linear combination of monomials in the $f_j$'s at least one of which has exponent $\geq N$. It follows that $S_q m_j = 0$ for $q$ big enough and all finitely many $j$'s, so $S_q M = 0$ for all big $q$. By Proposition 1.1(ii), this forces $M_n = 0$ for all big $n$. ■

## 2. Proof of Proposition 1.1

First we prove (i). The quotient map $\pi : S(d_1) \oplus \cdots \oplus S(d_r) \twoheadrightarrow M$ is a map of graded modules by design, so it is surjective in each degree. Hence, (i) reduces to the case of $S(d)$ for any $d \in \mathbf{Z}$. This has no terms in degree below $-d$ (since $S$ is $\mathbf{N}$-graded), and $S(d)_n = S_{d+n}$, so to prove (i) it remains to check that each $S_m$ is $S_0$-finite. As we discussed at the start, $S$ is a quotient of $S_0[X_1, \dots, X_N]$ with $X_j$ homogeneous of degree $\delta_j$. Thus, $S_m$ is a quotient of $S_0[X_1, \dots, X_N]_m$. Since each $X_j$ has *positive* degree, there are only finitely many monomials with any given degree $m \geq 0$, and those clearly span the degree-$m$ part as an $S_0$-module. This finishes the proof of (i).

Now turn to (ii). As in the proof of (i), we can reduce to the case $M = S(d)$ for $d \in \mathbf{Z}$. Shifting all degrees by $-d$ is harmless, so we can assume $M = S$. Say $S$ is generated over $S_0$ by homogeneous elements $f_1, \dots, f_N$ with positive degrees $\delta_1, \dots, \delta_N$. Consider $d_0$ that is a positive multiple of every $\delta_j$. Then we claim $S_{d_0} S_n = S_{d_0+n}$ for all sufficiently large $n > 0$. For any $m > 0$, $S_m$ is the $S_0$-linear span of monomials $f_1^{e_1} \cdots f_N^{e_N}$ with degree $m$. Taking $m$ big enough forces at least one $e_j$ to be as big as we wish, so in particular $e_j > d_0/\delta_j$ for some $j$. Hence, we can factor out the term $f_j^{d_0/\delta_j}$ with degree $d_0$. This completes the proof of (ii).

To prove (iii), by (i) we can drop all terms with $q$ below whatever bound we wish, which is to say we can replace $\oplus_{q \in \mathbf{Z}} M_{dq+r}$ with the direct sum in which $q$ is required to be as large as we wish to specify. Take $q > d_0 + n_0$ with $d_0$ as in (ii) and $n_0$ big enough as for $n$ in (ii).

Thus, by (ii),

$$M_{dq+r} = M_{d(d_0+n_0)+d(q-(d_0+n_0))+r} = S_{dd_0} M_{dn_0+d(q-(d_0+n_0))+r}.$$

This can keep being iterated if $q-(d_0+n_0) > d_0+n_0$, so we eventually arrive at an equality

$$M_{dq+r} = S_{dm} M_{dn_0+de+r}$$

where $0 \leq e \leq d_0+n_0$ and $m \in \mathbf{N}$. There are only finitely many possibilities for $dn_0+de+r$, so by (i) we arrive at a finite $S^{(d)}$-module generating set for $M^{(d)}$.

Finally, we consider (iv). We first discuss a general fact: for an $\mathbf{N}$-graded ring $T$ to be finitely generated as an algebra over its degree-0 part, it is necessary and sufficient that $T_+$ is a finitely generated ideal in $T$. The necessity is obvious (the algebra generators can be taken to be homogeneous of positive degree by decomposing some algebra generating set into finite sums of homogeneous elements, and those elements generate $T_+$ as an ideal). For sufficiency, we can arrange that the set $t_1, \dots, t_h$ of generators of $T_+$ as an ideal consists of homeogeneous elements with *positive* degree, so by degree-chasing with homogeneous elements it is easy to see by degree-induction that the same $t_j$'s are $T_0$-algebra generators of $T$.

Returning to our situation of interest, by taking $T$ to be $S^{(d)}$ above we see that to show the $\mathbf{N}$-graded $S^{(d)}$ is finitely generated over $S_0$ it is equivalent to showing $(S^{(d)})_+$ is finitely generated as an *ideal* in $S^{(d)}$. But $(S^{(d)})_+ = (S_+)^{(d)}$ (where $S_+$ is regarded as a graded $S$-module), and the original hypothesis that $S$ is finitely generated over $S_0$ implies (by taking $T = S$ above) that $S_+$ is a finitely generated ideal in $S$, which is to say it is a *finitely generated $S$-module*. We can then apply (iii) with $M = S_+$ and $r = 0$ to conclude that $(S_+)^{(d)}$ $(= (S^{(d)})_+)$ is a finitely generated $S^{(d)}$-module, and that exactly says this ideal of $S^{(d)}$ is finitely generated, as desired. Thus, the first assertion in (iv) is proved.

To complete the proof of (iv), it remains to find $d > 0$ so that $S^{(d)}$ is generated by $S_d$ over $S_0$. We can take $d$ to be $d_0$ as in (ii) for $M = S$.