Math 216A. Codimension

1. Main result and some interesting examples

Let k be a field, and A a domain finitely generated k-algebra. The dimension theory of A is linked to the structure of its fraction field $Q(A)$ in the sense that $\dim A = \mathrm{trdeg}_k(Q(A))$. In particular, all strictly increasing chains of irreducible closed subsets of $\mathrm{Spec}(A)$ have length bounded by $1 + \mathrm{trdeg}_k(Q(A))$, with some such chain achieving this maximal length. To make a sufficiently robust geometric theory of dimension, we need the following result (to be proved in the next section).

Theorem 1.1. *Let $Z = \mathrm{Spec}(A)$ for a domain A finitely generated over a field k. For every maximal chain of irreducible closed sets*

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Z$$

(so Z_0 is a point, by maximality), necessarily $n = \dim Z$.

In this theorem, "maximal" means (since Z_0 is a point and Z is irreducible) that the chain cannot be made longer by inserting an irreducible closed set strictly between some Z_i and Z_{i+1}. Since $\dim Z_0 = 0$ and $\dim Z_{i+1} > \dim Z_i$ for all i (as for any strict inclusion between irreducible closed subsets of Z), we have

$$\dim Z_i = \sum_{1 \leq j \leq i} (\dim Z_j - \dim Z_{j-1})$$

with each of the i differences an integer ≥ 1. But for $i = n$ we have $\dim Z_n = \dim Z = n$, so each of the differences $\dim Z_j - \dim Z_{j-1}$ must be exactly 1 and hence $\dim Z_i = i$ for all i.

In other words, for a maximal chain of irreducible closed sets every jump involves an increase of exactly 1 in the dimension. For example, in a 3-dimensional Z any maximal chain of irreducible closed subsets must be a closed point in an irreducible closed "curve" in an irreducible closed "surface" in Z. This has the following important corollary.

Corollary 1.2. *Let A' be a domain finitely generated over k with dimension d' and let $Z' = \mathrm{Spec}(A')$. If $Z = V(P) = \mathrm{Spec}(A'/P) \subseteq Z'$ is an irreducible closed subset of dimension $d \leq d'$ then every maximal chain of irreducible closed sets beginning at Z and ending at Z' has the form*

$$Z = Z_d \subsetneq \cdots \subsetneq Z_{d'} = Z'$$

with $\dim Z_i = i$ for every $d \leq i \leq d'$.

Proof. By maximality of such a chain, if we append on the left a maximal chain contained in Z then we get a maximal chain in Z', so by the Theorem applied to Z' this new chain (which begins at a point) must have $1 + \dim Z'$ terms and $\dim Z_i = i$ for all i. ■

In view of this corollary, we have a good notion of *codimension* in the irreducible case:

Definition 1.3. Let $Z' = \mathrm{Spec}(A')$ for a domain A' finitely generated over a field k. For an irreducible closed subset $Z \subseteq Z'$, the *codimension* $c = \mathrm{codim}_{Z'}(Z)$ of Z in Z' is the unique integer c such that every maximal chain of irreducible closed sets beginning at Z and ending at Z' has $c + 1$ terms. Equivalently, $c = \dim Z' - \dim Z$.

There is a reasonable definition of codimension without irreducibility hypotheses (i.e., allowing Z or Z' to be reducible), but it is not as geometrically significant as in the irreducible case, so we won't discuss it.

We end this introductory section with some instructive examples over an algebraically closed field k, working with MaxSpec instead of Spec for more direct contact with "geometric intuition" (though in such cases there is no difference between working with Spec or MaxSpec for questions about chains of irreducible closed subsets).

Inspired by linear algebra, it is natural to wonder if we can define codimension $c = \mathrm{codim}_{Z'}(Z)$ in terms of "minimal number of equations needed to cut out Z inside Z'". This can be interpreted in two reasonable ways. Since $k[Z] = k[Z']/J$ for a radical ideal J, we can consider the minimal number of generators of the ideal J *or* the minimal number of generators of *some* ideal $I \subset k[Z']$ such that $\mathrm{rad}(I) = J$. This latter condition is a weaker requirement (as we do not specify which I to use), but even for this it turns out that working with the number of equations does *not* give the right notion in general.

The problem is that we are working too globally. It turns out that in a suitable "local" sense (in the Zariski topology) one can always find a set of c "local equations" that define Z as a subset of Z' near an arbitrary chosen point $z \in Z$, but the proof rests on much deeper work in the dimension theory of local noetherian rings. For cutting out the entirety of Z in Z', there are counterexamples if we try to use only c global equations. We now describe such a counterexample, but we omit the justification (which require techniques in commutative algebra beyond the level of this course). This counterexample (due to Hartshorne) works in any characteristic, and involves a surface in 4-space. Roughly speaking, we consider a surface S obtained from the plane by identifying the points $(0, 0)$ and $(0, 1)$. More rigorously:

Example 1.4. Consider the k-subalgebra

$$A = \{f \in k[t, u] \mid f(0, 0) = f(0, 1)\} \subset k[t, u].$$

(The equation defining A corresponds to the geometric idea of identifying the points $(0, 0)$ and $(0, 1)$.) I claim that A is the k-subalgebra generated by 4 elements:

$$t, tu, u(u-1) = u^2 - u, u^2(u-1) = u^3 - u^2.$$

Geometrically, this means that A is the coordinate ring of a closed set in affine 4-space.

Clearly the 4 indicated elements of $k[t, u]$ lie in A. To prove that they generate A as a k-algebra, consider an arbitrary element $f \in A$. Since any $n > 1$ has the form $2a + 3b$ with integers $a, b \geq 0$, we can use the elements $u^2 - u$ and $u^3 - u^2$ in our list to write f in the form

$$f = h(t, u^2 - u, u^3 - u^2) + ug(t)$$

for some $h \in k[x, y, z]$. Likewise, $ug(t) = cu + tuG(t)$ for some $c \in k$ and $G \in k[t]$. Hence, we have expressed f as an element of $k[t, tu, u(u-1), u^2(u-1)]$ up to adding an element of the form cu. But $cu \in A$ precisely when $f \in A$ (as $t, tu, u(u-1), u^2(u-1) \in A$), and $cu \in A$ if and only if $c = 0$, so $f \in A$ if and only if $c = 0$. Thus, the asserted list of k-algebra generators of A really does work.

To summarize, we see that there is a surjective map $\pi : k[x, y, z, w] \twoheadrightarrow A$ via

$$x \mapsto t, y \mapsto u(u-1), z \mapsto tu, w \mapsto u^2(u-1).$$

The kernel $P := \ker \pi$ is a prime ideal (since the quotient A is a domain), and clearly

$$xw - yz, x^2y - z(z-x), y^3 - w(w-y) \in P. \tag{1}$$

In more geometric terms with $k = \overline{k}$, since the defining inclusion $A \hookrightarrow k[t,u]$ is injective, we see that the polynomial map $f : k^2 \to k^4$ defined by

$$f : (t,u) \mapsto (t, u(u-1), tu, u^2(u-1))$$

has image contained in $\underline{Z}(P)$ and *dense* in $\underline{Z}(P)$ (since $k[x,y,z,w]/P = A \hookrightarrow k[t,u]$ is injective). Since the injective map $k[x,y,z,w]/P = A \to k[t,u]$ is module-finite (e.g., $t \in A$ and $u^2 - u \in A$), the geometric map $k^2 \to \underline{Z}(P)$ is finite *surjective*, so $\dim \underline{Z}(P) = \dim k^2 = 2$. Thus, $Z := \underline{Z}(P)$ is an irreducible surface in k^4; it has codimension 2.

For $k = \overline{k}$, the elements in (1) vanish on Z, and they do cut out Z set-theoretically; i.e., their common zero locus in k^4 is $Z = f(k^2)$. Indeed, consider a point (x_0, y_0, z_0, w_0) that satisfies all three relations

$$x_0w_0 = y_0z_0, x_0^2y_0 = z_0(z_0 - x_0), y_0^3 = w_0(w_0 - y_0).$$

We seek $(t_0, u_0) \in k^2$ such that

$$(x_0, y_0, z_0, w_0) = (t_0, u_0(u_0-1), t_0u_0, u_0^2(u_0-1)).$$

The "easy" case is when $x_0 \neq 0$, in which case we define $u_0 = z_0/x_0$ and $t_0 = x_0$; this works since

$$u_0(u_0-1) = z_0(z_0-x_0)/x_0^2 = x_0^2y_0/x_0^2 = y_0, u_0^2(u_0-1) = u_0y_0 = z_0y_0/x_0 = w_0.$$

Suppose instead that $x_0 = 0$, so clearly $z_0 = 0$ and our point is $(0, y_0, 0, w_0)$ with $y_0^3 = w_0(y_0 - w_0)$. If $y_0 = 0$ then $w_0 = 0$ and so we can take $t_0 = u_0 = 0$. If $y_0 \neq 0$ then we can take $t_0 = 0$ and $u_0 = w_0/y_0$. This completes the proof that Z is the common zero locus of the elements in (1). (Note that we have not addressed whether or not these three elements of P in fact generate P. This is not necessary to know.)

To prove that Z cannot be the set of common zeros of a pair of polynomials in $k[x,y,z,w]$, one has to use deeper techniques from commutative algebra (related to completions and connectedness properties of Cohen-Macaulay rings). This is explained in Example 3.4.2 of Hartshorne's paper "Complete intersections and connectedness".

2. Proof of Theorem 1.1

We shall prove that if $Z = \mathrm{Spec}(A)$ for a domain A finitely generated over a field k then all maximal chains of irreducible closed subsets of Z have length $1 + \dim Z$. We argue by induction on $\dim Z$. If $\dim Z = 0$ then Z is a point and the result is clear. Thus, we may assume that the common value $\dim Z = \mathrm{trdeg}_k(Q(A))$ is positive. Since any maximal chain of irreducible closed subsets in Z ends with a chain $V = \mathrm{Spec}(A/P) \subsetneq Z$ where V is maximal among irreducible proper closed subsets of Z (i.e., P is a minimal nonzero prime), our task is equivalent to showing that if $V \subsetneq Z$ is a maximal irreducible proper closed subset of Z then $\dim(V) \stackrel{?}{=} \dim(Z) - 1$, as then we can apply dimension induction to conclude. It is equivalent to show that $\mathrm{trdeg}_k(Q(A/P)) = \mathrm{trdeg}_k(Q(A)) - 1$.

We first treat the special case $Z = \mathrm{Spec}(k[x_1, \ldots, x_d])$ with $d > 0$, and then we will use this case to handle the general case via the Noether normalization theorem. For such Z

we claim that the maximal proper irreducible closed subsets are precisely the irreducible hypersurfaces $V(f)$ for an irreducible $f \in k[x_1, \dots, x_d]$. Indeed, if P is any nonzero prime ideal of this polynomial ring then it contains a nonzero polynomial and thus (by primality) contains one of its irreducible factors f. That is, P contains (f), so the prime ideals (f) for irreducible $f \in k[x_1, \dots, x_d]$ are precisely the minimal nonzero primes of this polynomial ring. This yields the asserted description of the maximal irreducible proper closed subsets of $\mathrm{Spec}(k[x_1, \dots, x_d])$. Our task in this special case is to show that $\dim(V(f)) = d - 1$.

By relabeling variables we can assume that f involves x_d, so

$$f = a_n(x_1, \dots, x_{d-1})x_d^n + \cdots \in k[x_1, \dots, x_{d-1}][x_d]$$

with $n > 0$, the omitted terms of lower degree in x_d, and $a_n \in k[x_1, \dots, x_{d-1}]$ nonzero. Since f is irreducible and *involves* x_n, it is easy to see (check!) that f does not divide any nonzero element of $k[x_1, \dots, x_{d-1}]$. Thus, the natural map

$$k[x_1, \dots, x_{d-1}] \to k[x_1, \dots, x_d]/(f)$$

between domains is *injective*, yet the induced map of fraction fields

$$k(x_1, \dots, x_{d-1}) \to \mathrm{Frac}(k[x_1, \dots, x_d]/(f))$$

is finite algebraic since the element $x_d \in k[x_1, \dots, x_d]/(f)$ satisfies the positive-degree algebraic relation over $k(x_1, \dots, x_{d-1})$ given by the condition $f = 0$. Thus, by additivity of transcendence degree in towers of finitely generated field extensions,

$$\dim(V(f)) = \mathrm{trdeg}_k(\mathrm{Frac}(k[x_1, \dots, x_d]/(f))) = \mathrm{trdeg}_k(k(x_1, \dots, x_{d-1})) = d - 1,$$

as desired.

Now we consider the general case with $d = \dim(Z) > 0$. By Noether normalization there is a finite injective map $k[x_1, \dots, x_d] \hookrightarrow A$, hence the corresponding map

$$f : Z = \mathrm{Spec}(A) \to \mathrm{Spec}(k[x_1, \dots, x_d])$$

has dense image (as $\ker(f) = 0$) which is closed (due to the module-finiteness of the ring map) and thus full. That is, f is surjective and closed. The closed image $V' = f(V) \subseteq \mathrm{Spec}(k[x_1, \dots, x_d])$ is an irreducible closed subset and $V \to V'$ corresponds to a module-finite injection between the associated domains (as respective quotients of A and $k[x_1, \dots, x_d]$). Thus, the resulting equality of transcendence degrees over k for their fraction fields yields that $\dim V' = \dim V < \dim Z = d$, so $V' \neq k[x_1, \dots, x_d]$. It suffices to show that $\dim(V') = d-1$, so in view of the special case just treated above it suffices to show that V' is maximal as an irreducible proper closed subset of $k[x_1, \dots, x_d]$. Recall that V is maximal as an irreducible proper closed subset of Z, by hypothesis. It therefore suffices to apply the following general result to the module-finite injection $k[x_1, \dots, x_d] \hookrightarrow A$:

Proposition 2.1 (weak going-down theorem)**.** *Let $B \to C$ be a module-finite injection between domains with B integrally closed. If P is minimal as a nonzero prime ideal of C then $P \cap B$ is minimal as a nonzero prime ideal of B.*

Proof. By integrality of this ring extension, and the resulting incomparability of distinct primes of C over a common prime of B, any nonzero prime of C must lie over a nonzero prime of B. (This can also be seen directly by considering constant terms of suitable "minimal" polynomials.) Thus, $P \cap B \neq 0$. If there were a nonzero prime $\mathfrak{p}$ strictly contained in $P \cap B$ then by the going-down theorem (which applies since B is integrally closed!) there would be a prime ideal $P' \subset P$ in B lying over $\mathfrak{p}$. Thus, $P' \neq 0$ and $P' \neq P$, contradicting the minimality hypothesis on P. ■

Remark 2.2. The idea behind the preceding proof can be expressed in another way in the classical setting (working over an algebraically closed field k and with MaxSpec, for simplicity) which illuminates the role of integral closedness. Rather than showing that a maximal proper irreducible closed set $V \subset Z$ maps onto a maximal proper irreducible closed set $V' \subset Z'$ (with Z and Z' irreducible affine algebraic sets over k), we can just as well try to show that if $V' \subsetneq W' \subset Z'$ is a strict containment between general irreducible closed sets in Z' and if V is an irreducible closed set of Z that lies over V' in the sense that it maps onto V' then is V contained in an irreducible closed set W that maps onto W' (so $V \subsetneq W \subseteq Z$ if W' lies strictly between V' and Z')? *This* formulation of the problem turns out to be false when $k[Z']$ is not integrally closed. We now give a counterexample, taking $k = \overline{k}$ for simplicity of exposition.

Consider $f : k^2 \to k^3$ defined by $(x, y) \mapsto (x(x-1), x^2(x-1), y)$. The image consists of $(u, v, y) \in k^3$ for which $v^2 - uv - u^3 = 0$ (as we see by setting $x = v/u$ when $u \neq 0$), so for

$$Z' := \{(u, v, y) \in k^3 \mid v^2 - uv - u^3 = 0\}$$

it is easy to check that Z' is an irreducible surface and $f : k^2 \to Z'$ is a finite map (since $x^2 - x$ and y lie in the coordinate ring $k[Z'] \subset k[x, y]$). In fact, the module-finite inclusion $k[Z'] \hookrightarrow k[x, y]$ induces an equality of fraction fields since $y \in k[Z']$ and $x = v/u$ with $u, v \in k[Z']$, so $k[x, y]$ is the integral closure of $k[Z']$ in its fraction field and it is strictly larger (e.g., $x \notin k[Z']$). That is, $k[Z']$ is not integrally closed.

Geometrically, f carries both lines $L_0 = \{x = 0\}$ and $L_1 = \{x = 1\}$ onto the y-axis $L = \{u = v = 0\} \subset Z'$ in k^3, with $f^{-1}(L) = L_0 \cup L_1$. Away from L the restricted map $k^2 - (L_0 \cup L_1) \to Z' - L$ is an isomorphism between these basic affine open sets (i.e., the associated map of coordinate rings $k[x, y]_{x(x-1)} \to k[Z']_{x-y}$ is an isomorphism), so we visualize Z' as the result of making the plane k^2 pass through itself along a single line L.

Consider the diagonal line $\Delta = \{x = y\}$ in k^2 which meets $L_0 = \{x = 0\}$ in $(0, 0)$ and meets $L_1 = \{x = 1\}$ in $(1, 1)$. The image $C' = f(\Delta) \subset Z'$ is an irreducible closed set in Z' of dimension 1 that meets the common image L of L_0 and L_1 in the points $P = (0, 0, 0)$ and $Q = (0, 0, 1)$. Visually, C' is a curve in Z' that "wraps around" the surface Z', passing through the line of singularities L at the points P and Q. In particular, the preimage $f^{-1}(C') = \Delta \cup \{(1, 0)\} \cup \{(0, 1)\}$ is a disjoint union of the diagonal Δ and two isolated points $(1, 0)$ and $(0, 1)$. Thus, if we consider the irreducible closed set $V' = \{P\}$ in C' and choose the irreducible closed set $V = \{(1, 0)\}$ over V' then there is *no* irreducible closed set C in $Z = k^2$ that contains V' and maps onto the irreducible closed set C' that contains V'. Indeed, since C' is irreducible of dimension 1 then any such C would have to be irreducible of dimension 1 and yet lie in $f^{-1}(C')$ which is a disjoint union of Δ and two isolated points. That is, the only possibility for C is Δ, yet this does not contain $V = \{(1, 0)\}$!