Math 216A. Preservation of properties of morphisms under base change

1. Introduction

For a scheme S, an S-map $f : X \to Y$, and an S-scheme S', we get an associated S'-map $f' : X' \to Y'$ via base change. That is,

$$f' = f \times \mathrm{id}_{S'} : X \times_S S' \to Y \times_S S'.$$

In this handout, we show for many properties $\mathbf{P}$ of scheme morphisms that if f satisfies $\mathbf{P}$ then so does f'. (There will be many more properties to come later that will also be preserved in this way.)

It is extremely important to also be able to go in reverse, showing that f inherits properties verified for f', when $S' \to S$ is a reasonably "nice" type of morphism (generally involving various properties related to flatness in commutative algebra); that is the subject of Grothendieck's "descent theory", a fundamental tool in many constructions and proofs at a more advanced level (also subsuming Galois theory and gluing into a common framework). It is a vast generalization of the method in linear algebra for proving a result after applying an extension of the field (so as to acquire eigenvalues or for other reasons). We won't say anything about descent theory here.

Before we commence with the main discussion we explain why the role of S is essentially irrelevant. The point is that, in the notation above, the commutative diagram (of natural maps)

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longleftarrow & Y' \end{array}$$

is itself Cartesian (i.e., $X' \to X \times_Y Y'$ is an isomorphism), so one loses nothing by focusing on the case $S = Y$ (with $S' = Y'$)! To see that $X' \to X \times_Y Y'$ is an isomorphism, we use the "associativity" property of fiber products as discussed in class: this associativity says

$$X \times_Y Y' = X \times_Y (Y \times_S S') = (X \times_Y Y) \times_S S',$$

yet the Y-morphism $\mathrm{pr}_X : X \times_Y Y \to X$ is an isomorphism (think above the universal property of this fiber product, or evaluate both sides on a general Y-scheme and use Yoneda's Lemma), so $X \times_Y Y' \simeq X \times_S S' = X'$, and this resulting isomorphism $X \times_Y Y' \simeq X'$ is exactly an inverse to the natural map $X' \to X \times_Y Y'$ (check this, again most readily via Yoneda's Lemma).

2. Main result

A map $f : X \to Y$ of schemes is called *flat* if the induced map of local rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat for all $x \in X$. It is equivalent that for any affine open $\mathrm{Spec}(B) \subset Y$ and $\mathrm{Spec}(A) \subset f^{-1}(\mathrm{Spec}\, B)$, the ring map $B \to A$ is flat (or equivalently that for some affine open cover $\{\mathrm{Spec}(B_i)\}$ of Y and affne open cover $\{\mathrm{Spec}(A_{ij})\}$ of each $f^{-1}(\mathrm{Spec}\, B_i)$, the ring maps $B_i \to A_{ij}$ are flat) due to:

Lemma 2.1. *A map of affine schemes* $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ *is flat in the sense defined above if and only if* A *is* B*-flat.*

Proof. Letting $\varphi : B \to A$ correspond to f, by definition f is flat when $B_{\varphi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$ is flat for every prime ideal $\mathfrak{p}$ of A. First assume φ is flat. Since localization preserves flatness, the localization $B_{\varphi^{-1}(\mathfrak{p})} \to A_{\varphi^{-1}(\mathfrak{p})}$ of φ (where the target is the localization of A at the multiplicative set $B - \varphi^{-1}(\mathfrak{p}) \subset \varphi^{-1}(A - \mathfrak{p})$) is flat. But localization $R \to S^{-1}R$ of any ring at a multiplicative set is flat, so the ring map $A_{\varphi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$ is flat, and hence the composition

$$B_{\varphi^{-1}(\mathfrak{p})} \to A_{\varphi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$$

is flat. This compsition is the stalk map induced by f, so since $\mathfrak{p}$ is any point in $\mathrm{Spec}(A)$ we conclude that f is flat.

Conversely, suppose f is flat. We want to show that φ is flat, which is to say that $A \otimes_B (\cdot)$ is an exact functor on B-modules. But to check exactness it is the same for exactness to hold after localizing at every prime ideal $\mathfrak{p}$ of A, and applying such localization yields the functor $A_{\mathfrak{p}} \otimes_B (\cdot)$ on B-modules. For any B-module M we have naturally

$$A_{\mathfrak{p}} \otimes_B M \simeq A_{\mathfrak{p}} \otimes_{B_{\varphi^{-1}(\mathfrak{p})}} (B_{\varphi^{-1}(\mathfrak{p})} \otimes_B M) \simeq A_{\mathfrak{p}} \otimes_{B_{\varphi^{-1}(\mathfrak{p})}} M_{\varphi^{-1}(\mathfrak{p})}.$$

This is an exact functor in M because it is a composition of exact functors: the exact localization functor $M \rightsquigarrow M_{\varphi^{-1}(\mathfrak{p})}$ from B-modules to $B_{\varphi^{-1}(\mathfrak{p})}$-modules, and the functor $A_{\mathfrak{p}} \otimes_{B_{\varphi^{-1}(\mathfrak{p})}} (\cdot)$ on $B_{\varphi^{-1}(\mathfrak{p})}$-modules which is exact by the assumed flatness of f. So we conclude that φ is flat as desired. ■

Note that a compostion of flat maps is flat: this holds either by the definition in terms of stalks or in terms of the characterization with compatible affine open subschemes, in either case because of the algebraic fact that a composition of flat ring maps is flat.

Here is a useful variant on finiteness for morphisms:

Definition 2.2. A map of schemes $f : X \to Y$ is *integral* if it is affine and for some affine open cover $\{\mathrm{Spec}(B_i)\}$ of Y the affine preimages $\mathrm{Spec}(A_i) = f^{-1}(\mathrm{Spec}(B_i))$ have A_i integral over B_i (i.e., $B_i \to A_i$ is integral).

As usual, we want to know that if f is integral then for *every* open affine $\mathrm{Spec}(B) \subset Y$ the affine preimage $\mathrm{Spec}(A) = f^{-1}(\mathrm{Spec}(B))$ has A integral over B. By the Nike trick, this amounts to showing that if $B \to A$ is a ring map and $B_{b_j} \to A_{b_j}$ is integral for b_j's in B that generate the unit ideal then A is integral over B.

Integrality amounts to each $a \in A$ generating a B-finite subalgebra of A, so to show A is integral over B we want the B-subalgebra $B[a] \subset A$ to be module-finite for each $a \in A$. That is, we want $h : \mathrm{Spec}(B[a]) \to \mathrm{Spec}(B)$ to be finite. But $B[a]_{b_j} = B_{b_j}[a] \subset A_{b_j}$, so h becomes finite over the constituents $\{\mathrm{Spec}(B_{b_j})\}$ of an open cover of $\mathrm{Spec}(B)$, and hence h is finite, as desired.

Here is the main result.

Theorem 2.3. *Let* $\mathbf{P}$ *be any of the following properties of scheme morphisms* $f : X \to Y$*.*

(i) *open immersion,*
(ii) *closed immersion,*
(iii) *affine,*

(iv) *finite,*
(v) *integral,*
(vi) *locally of finite type,*
(vii) *quasi-compact,*
(viii) *finite type,*
(ix) *quasi-finite,*
(x) *flat,*
(xi) *surjective.*

If Y' is any Y-scheme and f satisfies $\mathbf{P}$ then so does the base change $f' : X' = X \times_Y Y' \to Y'$.

By the discussion in §1 concerning the irrelevance of S, this result establishes the preservation of all properties (i)–(xi) in the base change situation over S considered in §1.

Proof. Let $p : Y' \to Y$ be the "structure map" which makes Y' a Y-scheme. First we treat (i), so assume $X \to Y$ is an open subscheme. The functorial characterization of $X \times_Y Y'$ identifies it with $p^{-1}(X)$ since both represent the same functor on Y-schemes (compatibly with the evident map $\mathrm{pr}_2 : X \times_Y Y' \to p^{-1}(X)$). In effect, this is similar to the arguments with open immersions in the construction of fiber products. The upshot is that the Y'-scheme $X \times_Y Y'$ is identified with the open subscheme $p^{-1}(X) \subset Y'$. Hence, f' is the open immersion $p^{-1}(X) \hookrightarrow Y'$.

Next, we turn to (ii). Checking whether a map of ringed spaces $h : T \to S$ is a closed immersion is "local on the target" (i.e., it is equivalent to check for each $h^{-1}(U) \to U$ for U varying through an open cover of S), so we can work locally on Y (replacing X and Y' with the respective preimages in each of open subschemes of Y) to reduce to the case where Y is affine, and then likewise reduce to the case where Y' is affine. Now we have $Y = \mathrm{Spec}(B)$ and $Y' = \mathrm{Spec}(B')$ for some ring map $B \to B'$. But since $X \to Y$ is a closed immersion, by Exercise 3.11(b) from HW6 we have $X = \mathrm{Spec}(B/I)$ for some ideal I of B. Then $X' = X \times_Y Y' = \mathrm{Spec}((B/I) \otimes_B B') = \mathrm{Spec}(B'/IB')$ with $\mathrm{pr}_2 : X' \to Y'$ identified with the map $\mathrm{Spec}(B'/IB') \to \mathrm{Spec}(B')$ corresponding to the surjection $B' \to B'/IB'$ (make sure you understand why this identification of pr_2 is correct!). Hence, this is a closed immersion.

The proof of (iii) goes similarly to (ii) but is even simpler: we similarly reduce to the case when Y and Y' are affine, and then X is affine by Exercise A in HW6 (since $X \to Y$ is affine with target that is affine). Then $X \times_Y Y'$ is affine by the construction of fiber products in the affine setting, so $X \times_Y Y' \to Y'$ is affine, as desired.

The proof of (iv) goes similarly to (iii), with the extra input (from various equivalent characterizations of finiteness) that if B is a ring and A and B' are B-algebras with A module-finite over B then $A \otimes_B B'$ is clearly module-finite over B' (generators being $\{a_i \otimes 1\}$ for a finite set $\{a_i\}$ of B-module generators of A). The proof of (v) is identical with "module-finite" replaced by "integral" since $A \otimes_B B'$ is generated as a B'-algebra by the elements $a \otimes 1$ that are certainly integral over B'. The proof of (vi) is essentially the same: we reduce to the case where $Y = \mathrm{Spec}(B)$ and $Y' = \mathrm{Spec}(B')$ are affine, and if $\{\mathrm{Spec}(A_i)\}$ is an affine open cover of X (so each A_i is finitely generated as a B-algebra, due to f being locally of finite type) then $\{\mathrm{Spec}(A_i \otimes_B B')\}$ is an affine open cover of $X \times_Y Y'$. But each $A_i \otimes_B B'$ is finitely generated as a B'-algebra (with algebra generators as such given by $a_{ij} \otimes 1$ for a finite set $\{a_{ij}\}$ of B-algebra generators of A_i), so indeed f' is locally of finite type.

For quasi-compactness, as usual we can reduce to the case that $Y = \mathrm{Spec}(B)$ and $Y' = \mathrm{Spec}(B')$ are affine. By quasi-compactness of f, X is covered by a *finite* collection of affine open subschemes $\mathrm{Spec}(A_i)$, and then $X' = X \times_Y Y'$ is covered by the finite collection of affine open subschemes $\mathrm{Spec}(A_i \otimes_B B')$ by design of fiber products. Hence, X' is quasi-compact, so f' with affine target Y' is quasi-compact, settling (vi). By defintition of "finite type" as the synthesis of (v) and (vi), we get (vii).

For quasi-finiteness (which means, by the correction of the definition in [H], "finite type with finite fibers"), since "finite type" has been handled we just have to check preservation under base change for having finite fibers when $f : X \to Y$ is given to also be of finite type. To analyze the fiber of $f' : X' \to Y'$ at a point $y' \in Y'$, we use that the topological fiber ${f'}^{-1}(y')$ is identified with the fiber product $X' \times_{Y'} \mathrm{Spec}(k(y'))$ and exploit the "associativity" of fiber products:

$$\begin{aligned} X' \times_{Y'} \mathrm{Spec}(k(y')) = (X \times_Y Y') \times_{Y'} \mathrm{Spec}(k(y')) &= X \times_Y (Y' \times_{Y'} \mathrm{Spec}(k(y'))) \\ &= X \times_Y \mathrm{Spec}(k(y')). \end{aligned}$$

Since $Y' \to Y$ carries y' to y, so the composite map $\mathrm{Spec}(k(y')) \to Y' \to Y$ coincides with the composite map $\mathrm{Spec}(k(y')) \to \mathrm{Spec}(k(y)) \to Y$, we can run the associativity of fiber products in reverse to obtain

$$\begin{aligned} X \times_Y \mathrm{Spec}(k(y')) &= X \times_Y (\mathrm{Spec}(k(y)) \times_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(y'))) \\ &= (X \times_Y \mathrm{Spec}(k(y))) \times_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(y')) \\ &= X_y \times_{k(y)} k(y'). \end{aligned}$$

This says *a fiber of a base change is the scalar extension of the original fiber along the extension of residue fields.* By hypothesis the $k(y)$-scheme X_y has finite underlying space, and it is finite type over $k(y)$ because $X \to Y$ is finite type. So to settle (ix) we use:

Lemma 2.4. *If Z is a scheme of finite type over a field k then $|Z|$ is finite if and only if Z is k-finite (i.e., $Z = \mathrm{Spec}(A)$ for a k-finite algebra A). In particular, for any field extension k'/k, the base change $Z \times_k k'$ is k'-finite (and so has finite underlying space).*

Proof. It is clear that if Z is k-finite then $|Z|$ is finite (a k-algebra with finite dimension as a k-vector space is an artinian ring and so has finite Spec since every artinian ring is a direct product of finitely many artin local rings).

Now assume $|Z|$ is finite. We want to deduce that Z is k-finite. First we treat the case when Z is also assumed to be affine. Then $Z = \mathrm{Spec}(A)$ for a finite type k-algebra A, and if its dimension is $d \geq 0$ then Noether normalization (which is valid over any field, even finite fields) expresses A as a module-finite extension of $k[T_1, \dots, T_d]$. Then $\mathrm{Spec}(A) \to \mathbf{A}_k^d$ is surjective, yet $\mathrm{Spec}(A)$ has finite underlying space, so this rules out the possibility $d > 0$ (since if $d > 0$ then $\mathbf{A}_k^d$ maps onto $\mathbf{A}_k^1$ and $k[T]$ has infinitely many maximal ideals by a variant of Euclid's argument for primes in $\mathbf{Z}$). So A is 0-dimensional. But 0-dimensional noetherian rings are the same as artinian rings, and an artinian ring is a finite direct product of artin local rings. That is, $A = \prod_{i=1}^m A_i$ for artin local A_i. The residue field of each A_i is k-finite (by the Nullstellensatz, since each A_i is finite type over k), and A_i has finite length over itself (as for any artin local ring), so we conclude that $\dim_k A_i < \infty$! Hence, A is k-finite.

To summarize: if Z is *affine* of finite type over k and $|Z|$ is finite then Z is k-finite. But then $|Z|$ is even a discrete space (as $Z = \mathrm{Spec}(A)$ with A a finite direct product of artin local rings, each of which has a 1-point spectrum). In general, without assuming Z to be affine (just that $|Z|$ is finite), we can at least find a finite open cover by affines $U_1, \dots, U_m$. Of course each $|U_j|$ is finite, so by the settled affine case each U_j is k-finite with underlying discrete space. It follows that $|Z|$ is discrete (i.e., all points are open), since discreteness can be checked on the constituents of an open cover, so Z is a finite disjoint union of 1-point open subspaces which must themselved by affine (as affines are a base of opens for *any* scheme). By the settled affine case, those 1-point open subspaces are k-finite, say $\mathrm{Spec}(A_i)$ for k-finite artin local A_i, so as a ringed space over k we have

$$Z = \coprod \mathrm{Spec}(A_i) = \mathrm{Spec}(\prod A_i)$$

with $\prod A_i$ visibly k-finite (that $\mathrm{Spec}(A) \coprod \mathrm{Spec}(B) = \mathrm{Spec}(A \times B)$ for any rings A and B is a pleasant exercise: for the elements $e = (1,0)$ and $e' = (0,1)$ observe that $A \times B \to A$ becomes an isomorphism upon inverting e and $A \times B \to B$ becomes an isomorphism upon inverting e', and these basic affine opens in $\mathrm{Spec}(A \times B)$ are *disjoint* since $e + e' = 1$). ■

To prove (x), we first note that flatness is Zariski-local on the source and target due to the various ways we have to think about flatness. Hence, we may argue as in the treatment of (v) to reduce to the case where X, Y, Y' are all affine, say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Y' = \mathrm{Spec}(B')$. Hence, the Y'-scheme $X' = X \times_Y Y'$ is $\mathrm{Spec}(A \otimes_B B')$ as a B'-scheme in the evident manner. By hypothesis A is B-flat, and we want to show that $A \otimes_B B'$ is B'-flat. But rather generally, if M is any flat B-module (such as $M = A$), then B'-module $M \otimes_B B'$ is B'-flat (since for any B'-module N' we have $(M \otimes_B B') \otimes_{B'} N' = M \otimes_B N'$, which is visibly exact in N'). This settles (ix).

Finally, we have to prove (xi). Given that $f : X \to Y$ is surjective, we want to show that $X \times_Y Y' \to Y'$ is surjective. For the moment drop any hypotheses on f. It is generally hard to describe the underlying topological space $|X \times_Y Y'|$. But via the maps $X \times_Y Y' \rightrightarrows X, Y'$ we get maps of topological spaces $|X \times_Y Y'| \rightrightarrows |X|, |Y'|$ which agree when composed with the maps $|X| \to |Y|$ and $|Y'| \to |Y|$ (since the compositions of $X \times_Y Y' \rightrightarrows X, Y'$ with $X \to Y$ and $Y' \to Y$ coincide by the definition of the fiber product over Y). Hence, without any hypotheses on f, we have a natural map of sets

$$\pi : |X \times_Y Y'| \to |X| \times_{|Y|} |Y'|$$

(where the right side is the set-theoretic fiber product: the subset of points $(x, y') \in |X| \times |Y'|$ for which x and y' lie over the same $y \in |Y|$).

Lemma 2.5 below gives that π is *always* surjective (it generally is not injective, as we shall discuss at length shortly). Hence, to show that $f' : X' \to Y'$ is surjective when f is surjective, the factorization of $|f'|$ as

$$|X'| \overset{\pi}{\twoheadrightarrow} |X| \times_{|Y|} |Y'| \overset{\mathrm{pr}_2}{\to} |Y'|$$

reduces our task to showing that the second step in this factorization is surjective when $|f|$ is surjective. But that is obvious: any $y' \in |Y'|$ has an image point $y \in |Y|$, and by surjectivity of f we have $y = f(x)$ for some $x \in |X|$, so (x, y') is a point in $|X| \times_{|Y|} |Y'|$ that is carried to y' via pr_2. ■

Lemma 2.5. *For any pair of scheme maps $X \to S$ and $Y \to S$, the natural map $\pi : |X \times_S Y| \to |X| \times_{|S|} |Y|$ is surjective.*

Proof. Pick any points $x \in |X|$ and $y \in |Y|$ over a common point $s \in |S|$. We seek a point $\xi \in X \times_S Y$ so that $\pi(\xi) = (x, y)$. Observe that $x \in X_s$ and $y \in Y_s$; i.e., these are points in the scheme-theoretic fibers, whose underlying topological spaces are the s-fibers of the maps $X \to S$ and $Y \to S$ (by Exercise 3.10(a) in HW7). Since $X_s \times_{\mathrm{Spec}(k(s))} Y_s = (X \times_S Y)_s$ (as discussed in class), it suffices to find ξ as a point in this latter s-fiber.

Hence, we are reduced to the case $S = \mathrm{Spec}(k)$ for a field k. In this case, our task is to show that the natural map

$$|X \times_k Y| \to |X| \times |Y|$$

is surjective. For $x \in |X|$ and $y \in |Y|$, we have scheme maps $f : \mathrm{Spec}(k(x)) \to X$ and $g : \mathrm{Spec}(k(y)) \to Y$ which hit exactly x and y. Then we have the fiber product map

$$f \times g : \mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k(y)) \to X \times_k Y.$$

If the source is non-empty then any point in it is carried to a point $\xi \in X \times_k Y$ that lies over $x \in |X|$ and $y \in |Y|$ since composing $f \times g$ with pr_1 and pr_2 yields composite maps respectively agreeing with the composite maps

$$\mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(x)) \to X,$$

$$\mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(y)) \to Y$$

(check this agreement with such composite maps!).

So finally our task is reduced to showing that $\mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k(y))$ is non-empty. This fiber product is the spectrum of the ring $k(x) \otimes_k k(y)$, so we just need this tensor product ring to be nonzero. But by the existence of bases for *arbitrary* k-vector spaces, we know that if V and W are any nonzero k-vector spaces whatsoever (such as $V = k(x)$ and $W = k(y)$) then $V \otimes_k W$ is nonzero! ■

In general the tensor product $k(x) \otimes_k k(y)$ is a gigantic k-algebra. For example, if $k(x) = k(T)$ and $k(y) = k(T)$ are rational function fields in 1 variable over k then by localization considerations $k(x) \otimes_k k(y) = S^{-1}k[T_1, T_2]$ where S is the multiplicative set of products $f(T_1)g(T_2)$ for nonzero $f, g \in k[T]$. This localization has spectrum with infinitely many prime ideals (corresponding to irreducible polynomials in $k[T_1, T_2]$ that involve both variables). So typically the fibers of the $|X \times_S Y| \to |X| \times_{|S|} |Y|$ are gigantic (in contrast with the setting of classical algebraic sets over an algebraically closed field k, which uses only k-valued points).

This explains why the definition of "quasi-finite" in [H, Ch. II, Exer. 3.5] (as "finite fibers", without the "finite type" condition we have required) is not a good definition: it is not preserved under base change, as the following example illustrates.

Example 2.6. For any field extension K/k that is not algebraic, the natural map $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ (which is not of finite type) has finite fibers, but it has a base change with infinite fibers. Namely, we pick $T \in K$ that is transcendental over k, and we claim that the base change by $\mathrm{Spec}(k(T)) \to \mathrm{Spec}(k)$ has infinite fibers. This base change is $\mathrm{Spec}(K \otimes_k k(T)) \to \mathrm{Spec}(k(T))$, which factors as

$$\mathrm{Spec}(K \otimes_k k(T)) \to \mathrm{Spec}(k(T) \otimes_k k(T)) \to \mathrm{Spec}(k(T))$$

whose first step is surjectve (being a base change of the surjection $\mathrm{Spec}(K) \to \mathrm{Spec}(k(T))$), so it is enough that the second map $\mathrm{pr}_2 : \mathrm{Spec}(k(T) \otimes_k k(T)) \to \mathrm{Spec}(k(T))$ has an infinite fiber. But we described $k(T) \otimes_k k(T)$ above and saw that its Spec is infinite.

3. Injectivity and base change

If $f : T \to S$ is an injective map of sets and $p : S' \to S$ is any map of sets, then the "base change" $f' : T' = T \times_S S' \to S'$ of f (which is just $\mathrm{pr}_2 : (t, s') \mapsto s'$) is also injective. Indeed, suppose $(t_1, s'_1), (t_2, s'_2) \in T'$ lie in the fiber over the same point $s' \in S'$ (so in particular $f(t_j) = p(s'_j)$ by definition of $T \times_S S'$ as a set). We have $s'_j = f(t_j, s'_j) = s'$, so $s'_1 = s'_2$, and hence $f(t_j) = p(s'_j) = p(s')$. But the resulting equality $f(t_1) = p(s') = f(t_2)$ forces $t_1 = t_2$ since f is injective, so we have shown $(t_1, s'_1) = (t_2, s'_2)$.

For schemes, the underlying space of a fiber product is usually not the fiber product of the underlying spaces (even as a set); we have seen that the natural map $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ is surjective, which in turn was a key ingredient in the proof that surjectivity is preserved by base change. But in contrast, injectivity on underlying spaces for a scheme morphism f can be lost under base change. Example 2.6 illustrated this in a case with f not of finite type, but such failure is very common even in reasonable situations such as maps between finite type schemes over a field k.

For example, consider $C \subset \mathbf{A}^2_k$ the integral curve defined by $y^2 = f(x)$ for a non-square monic irreducible $f \in k[x]$ with $\mathrm{char}(k) \neq 2$, and the projection $C \to \mathbf{A}^1_k$ to the x-axis. This has a 1-point fiber $\{c\}$ over each rational point $x = a \in k$ for which $f(a)$ is not a square in k (there are many such a in general), with that fiber $\mathrm{Spec}(k(c))$ corresponding to a separable quadratic extension k'/k. The base change $C_{k'} \to \mathbf{A}^1_{k'}$ has fiber over $x = a$ given by $\mathrm{Spec}(k(c)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') = \mathrm{Spec}(k' \otimes_k k') = \mathrm{Spec}(k') \coprod \mathrm{Spec}(k')$ since $k' \otimes_k k' \simeq k' \times k'$ via $a \otimes b \mapsto (ab, a\sigma(b))$ for any separable quadratic extension k'/k with nontrivial automorphism σ. Such splitting from separability is even more ubiquitous:

Example 3.1. Let K/k be an algebraic extension of fields for which $[K : k]_s > 1$ (e.g., a nontrivial finite separable extension). Then the base change of the natural map $f : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ by itself is $f' = \mathrm{pr}_2 : \mathrm{Spec}(K \otimes_k K) \to \mathrm{Spec}(K)$ corresponding to $b \mapsto 1 \otimes b$, and we claim that f' is never injective: the source always has more than one point. Indeed, let's first reduce to the case when K/k is a nontrivial finite separable extension.

We can pick a nontrivial separable subextension $F \subset K$ over k with $[F : k]$ finite, and then we have the factorization of f as $\mathrm{Spec}(K \otimes_k K) \to \mathrm{Spec}(F \otimes_k K) \to \mathrm{Spec}(K)$ with the first map surjective (since it is the base change of the surjection $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$), so it suffices to show the second map has a fiber with at least two points. But this second map is a base change of $\mathrm{pr}'_2 : \mathrm{Spec}(F \otimes_k F) \to \mathrm{Spec}(F)$, so if this has a fiber with at least two points then so does any base change of it (since in general $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ is surjective, applied to $\mathrm{Spec}(F \otimes_k F) \times_{\mathrm{Spec}(F)} \mathrm{Spec}(K)$). In this way it is enough to work with F in place of K, so now K/k is a nontrivial finite separable extension.

By the Primitive Element Theorem, there exists a primitive element $a \in K$ over k, so $K = k[x]/(h)$ for the monic separable minimal polynomial $h \in k[x]$ for a with degree $d > 1$. Then the K-algebra $K \otimes_k K$ (via the second tensor factor, say) is identified with $K[x]/(h)$ with $h = (x - a)g$ for g with degree $d - 1 > 0$ that doesn't vanish at a (as h is separable).

Hence, by the Chinese Remainder Theorem, as K-algebras we have $K[x]/(h) = K \times K[x]/(g)$, so $\mathrm{Spec}(K \otimes_k K) = \mathrm{Spec}(K) \coprod \mathrm{Spec}(K[x]/(g))$ has at least 2 points.

In general, if $f : X \to Y$ is a map of schemes and $x \in X$ is a point with image y, then when $k(x)/k(y)$ is either not algebraic (so Example 2.6 applies) or is algebraic but not purely inseparable (so Example 3.1 applies) then for the composite map $Y' = \mathrm{Spec}(k(x)) \to \mathrm{Spec}(k(y)) \to Y$ the base change $X' = X \times_Y Y' \to Y'$ has source with at least 2 points since $\mathrm{Spec}(k(x) \otimes_{k(y)} k(x))$ has at least 2 points (by Example 2.6 or 3.1) and the composite map

$$\mathrm{Spec}(k(x) \otimes_{k(y)} k(x)) = \mathrm{Spec}(k(x)) \times_{\mathrm{Spec}(k(y))} Y' \to X \times_Y Y'$$

is injective on underlying sets (since upon replacing X and Y with compatible affine opens around x and y we can describe the map ring-theoretically in terms of localization and quotients by ideals, each of which induce injective maps on Spec).

In other words, if $f : X \to Y$ is to be injective after any base change, a *necessary* condition is that not only must f be injective (the case of trivial base change) but also every residue field extension $k(x)/k(f(x))$ must be purely inseparable algebraic. This is also sufficient:

Theorem 3.2. *For a map of schemes $f : X \to Y$, the following are equivalent:*

(i) *For every $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ is injective.*
(ii) *The map f is injective and all residue field extensions $k(x)/k(f(x))$ are purely inseparable algebraic.*
(iii) *For any field L, the induced map on L-valued points $X(L) \to Y(L)$ is injective.*

Such maps f are called *universally injective*, or (following EGA) *radicial*. Open and closed immersions are evident examples, and a more interesting example is the normalization map for the cuspidal cubic ($\mathbf{A}^1_k \to \{y^2 = x^3\} \subset \mathbf{A}^2_k$ defined by $t \mapsto (t^2, t^3)$). By (i) and the "associativity" of fiber products, any base change of a radicial morphism is radicial.

Proof. We have already seen that (ii) is necessary for (i), which is to say (i) implies (ii). Also, (i) is necessary for (iii) (i.e., (iii) implies (i)) because if some base change $f' : X' \to Y'$ fails to be injective, say with x'_1, x'_2 over a common point $y' \in Y'$, then for any residue field L of the *nonzero* ring $k(x'_1) \otimes_{k(y')} k(x'_2)$ the two maps $\mathrm{Spec}(L) \to \mathrm{Spec}(k(x'_j)) \to X' \to X$ have the same composition with f (namely both yield $\mathrm{Spec}(L) \to \mathrm{Spec}(k(y')) \to Y' \to Y$), which is to say $X(L) \to Y(L)$ is not injective.

It remains to show that (ii) implies (iii). Suppose $h_1, h_2 : \mathrm{Spec}(L) \rightrightarrows X$ are two maps whose compositions with f yield the same map $\mathrm{Spec}(L) \to Y$. We want to show $h_1 = h_2$. The image points $x_1, x_2 \in X$ of the h_j's lie over the same point $y \in Y$ (why?), so by injectivity of f we have $x_1 = x_2$. Calling this point $x \in X$, each h_j factors as some $g_j : \mathrm{Spec}(L) \to \mathrm{Spec}(k(x))$ followed by $\mathrm{Spec}(k(x)) \to X$, and the compositions of each g_j with $\mathrm{Spec}(k(x)) \to \mathrm{Spec}(k(y))$ coincide (since that can be checked after composing with $\mathrm{Spec}(k(y)) \to Y$). To show $h_1 = h_2$ we just need to show that $g_1 = g_2$.

The g_j's correspond to maps of fields $k(x) \rightrightarrows L$ whose restriction to the subfield $k(y) \subset k(x)$ coincide. But $k(x)/k(y)$ is *purely inseparable*, so to conclude that $g_1 = g_2$ it suffices to show more generally that any map of fields $j : F \to L$ is uniquely determined by its restriction to a subfield $E \subset F$ over which F is purely inseparable algebraic. In characteristic 0 we have $E = F$, so there is nothing to do. In characteristic $p > 0$, every $\alpha \in F$ satisfies $\alpha^{p^n} \in E$

for some $n > 0$ (depending on α), so $j(\alpha)^{p^n} = j(\alpha^{p^n})$ is determined since $j|_E$ is specified and $\alpha^{p^n} \in E$. But p^nth roots in characteristic p are unique when they exist, so the element $j(\alpha) \in L$ is uniquely determined as well. ■

Since all residue fields on a $\mathbf{Q}$-scheme have characteristic 0, an injective map of $\mathbf{Q}$-schemes $f : X \to Y$ is universally injective if and only if the residue field extensions $k(x)/k(f(x))$ at *all* points $x \in X$ are trivial. (An instance of this once again is the normalization of the cuspidal cubic, which is radicial over all fields.) It is a pleasant exercise with denominator-chasing to show that if X and Y are of finite type over field k and $f : X \to Y$ is an injective k-map then f is radicial if the residue field condition in (ii) holds at the closed points $x \in X$. In particular, if k is algebraically closed then the following are equivalent: f is radicial, $X(k) \to Y(k)$ is injective, and f is injective.

Remark 3.3. Beware that the property of $f : X \to Y$ being a categorical monomorphism (i.e., the map $X(T) \to Y(T)$ on T-valued points is injective for all schemes T) is *far stronger* than being universally injective. This will ultimately involve the use of non-reduced schemes, even if X and Y are reduced.

To see what is going on, first note that $\mathrm{pr}_1, \mathrm{pr}_2 : X \times_Y X \rightrightarrows X$ have compositions with f that coincide (by the very definition of $X \times_Y X$). Thus, the monomorphism property for f would force those two projections to be the same map. Conversely, when those projections agree then f is a categorical monomorphism since any $x, x' \in X(T)$ yielding the same $y \in Y(T)$ give rise to a point $(x, x') \in X(T) \times_{Y(T)} X(T) = (X \times_Y X)(T)$ which the equal projections carry to $x, x' \in X(T)$, forcing $x = x'$.

When f is radicial, the maps $X \times_Y X \rightrightarrows X$ agree on field-valued points (by a calculation similar to what we just did using criterion (iii) in Theorem 3.2 – check!) and so they would agree as morphisms if $X \times_Y X$ were reduced, forcing f to be a categorical monomorphism. Hence, for a radicial map to fail to be a monomorphism of schemes it is *necessary* that $X \times_Y X$ is non-reduced.

We now check that the necessary and sufficient monomorphism criterion $\mathrm{pr}_1 = \mathrm{pr}_2$ fails for the radicial normalization $\mathbf{A}^1_k \to C$ of the cuspidal cubic. The two maps $\mathbf{A}^1_k \times_C \mathbf{A}^1_k \rightrightarrows \mathbf{A}^1_k$ correspond to the k-algebra maps

$$k[t] \rightrightarrows k[t] \otimes_{k[x,y]/(y^2 - x^3)} k[t] = k[t_1, t_2]/(t_1^2 - t_2^2, t_1^3 - t_2^3) =: R$$

defined by $t \mapsto t_1, t_2$. In R we have $t_1^2 = t_2^2$ and $t_1^3 = t_2^3$, so since every $n \geq 2$ has the form $2a + 3b$ with integers $a, b \geq 0$ we have $t_1^n = t_2^n$ in R for all $n \geq 2$. It follows that every element of R can be written as $g(t_1) + h(t_1)t_2$ for some $g, h \in k[x]$, and (check rigorously!) the non-uniqueness in such an expression is precisely the modifications $g - x^3 f, h + x^2 f$ for $f \in k[x]$ (corresponding to the polynomial identity $T_2(T_1^2 - T_2^2) - (T_1^3 - T_2^3) = -T_1^3 + T_1^2 T_2$), so imposing $\deg(h) \leq 1$ enforces uniqueness too. In particular, $t_1 \neq t_2$ in R. This shows the two maps $k[t] \rightrightarrows R$ are different.

Since the normalization map $\mathbf{A}^1_k \to C$ between reduced schemes is radicial but not a monomorphism, by theoretical reasons discussed above we know that $\mathbf{A}^1_k \times_C \mathbf{A}^1_k = \mathrm{Spec}(R)$ has to be non-reduced. Let's make such non-reducedness explicit by exhibiting a nonzero nilpotent element in R. The element $t_1 - t_2 \in R$ is nonzero, but

$$(t_1 - t_2)^3 = t_1^3 - 3t_1^2 t_2 + 3t_1 t_2^2 - t_2^3 = t_1^3 - 3t_2^3 + 3t_1^3 - t_2^3 = 4(t_1^3 - t_2^3) = 0.$$