Math 216A. Relation of classical theory to schemes

The aim of this handout is to explain the precise relationship between classical "varieties" over an algebraically closed field k and certain k-schemes. It is set up as a series a step-by-step exercises, to be done according to your interest (not for submission, but feel free to ask about it in office hours). This is a polished version of [H, Ch. II, Prop. 2.6] (which has unpleasant irreducibility hypotheses).

For a reduced k-algebra A of finite type, we define a locally ringed space $\mathrm{MaxSpec}(A)$ as follows. The underlying topological space is the set of maximal ideals of A with the usual Zariski topology (closed sets are those sets of maximal ideals which contain a given ideal), so we have the usual Nullstellensatz correspondence between closed sets and radical ideals in A. As for the sheaf, we can mimic the $\mathscr{B}$-sheaf construction, so $\mathrm{MaxSpec}(A)$ has the structure of a ringed space of k-algebras. This is intrinsic to the k-algebra A.

Define an *abstract algebraic set* to be a ringed space of k-algebras locally isomorphic (as a ringed space of k-algebras) to ones of the form $\mathrm{MaxSpec}(A)$ for reduced k-algebras A of finite type. Morphisms are as ringed spaces of k-algebras. These are automatically locally ringed and morphisms are also automatically as such since all residue fields are k in the evident manner. The objects isomorphic to $\mathrm{MaxSpec}(A)$ are affine algebraic sets expressed in a more intrinsic manner (no reference to an ambient affine space), and the notion of morphism in such cases agrees with the classical notion (based on our earlier work with morphisms in the classical setting). Our aim is to discuss the proof of the following result.

Theorem. *There is an equivalence of categories between abstract algebraic sets over k and reduced k-schemes which are locally of finite type.*

1. Let $i : X \to Y$ be a continuous map of topological spaces such that $U \rightsquigarrow i^{-1}(U)$ sets up a bijection between the set of open sets in Y and the set of open sets in X (Exercises 2 and 3 will give key examples of such a situation). Show that the functors $\mathscr{F} \rightsquigarrow i^{-1}(\mathscr{F})$, $\mathscr{G} \rightsquigarrow i_*(\mathscr{G})$ define an equivalence between the categories of sheaves of abelian groups (rings, sets, etc.) on X and on Y (i.e., construct natural isomorphisms of functors $f : i^{-1}i_* \simeq \mathrm{id}_X$ and $g : i_*i^{-1} \simeq \mathrm{id}_Y$ so that $i_*(f) = g \circ i_*$ and $i^{-1}(g) = f \circ i^{-1}$).

2. Let X be a reduced k-scheme which is locally of finite type. Define $T(X)$ to be the topological space consisting of the closed points in X (with the subspace topology), and let $i : T(X) \to X$ be the inclusion. Define $\mathscr{O}_{T(X)} = i^{-1}(\mathscr{O}_X)$. Show that Exercise 1 applies to the present setting and that $(T(X), \mathscr{O}_{T(X)})$ is an abstract algebraic set over k. If $X = \mathrm{Spec}(A)$, show that $T(X)$ is isomorphic to $\mathrm{MaxSpec}(A)$.

3. For any topological space Y, define the set $t(Y)$ to be the set of irreducible closed sets in Y (so $t(\emptyset) = \emptyset$). Define a *closed* set in $t(Y)$ to be a set of the form $t(Z)$ for a closed set Z in Y.

(i) Check that this makes $t(Y)$ a topological space, and t is a covariant functor from the category of topological spaces to itself. Moreover, if $f : U \to Y$ is an open embedding of topological spaces, check that $t(f) : t(U) \to t(Y)$ is also an open embedding (and likewise for closed embeddings).

(ii) Check that the canonical map $i : Y \to t(Y)$ defined by $i(y) = \overline{\{y\}}$ is continuous and that Exercise 1 applies to i.

(iii) Let Y be an abstract algebraic set. Show that $i : Y \to t(Y)$ gives a homeomorphism of Y onto the set of closed points in $t(Y)$. Define $\mathscr{O}_{t(Y)} = i_*(\mathscr{O}_Y)$, a sheaf of *reduced* k-algebras, so $t(Y)$ is now regarded as a ringed space of k-algebras. Prove that $t(Y)$ is a reduced k-scheme, locally of finite type over k. In fact, if Y is $\mathrm{MaxSpec}(A)$, then construct an isomorphism of locally ringed spaces of k-algebras $t(Y) \simeq \mathrm{Spec}(A)$.

4. Show that t and T are naturally *functors* between the categories of locally finite type reduced k-schemes and abstract algebraic sets over k. Construct explicit *natural* isomorphisms $F_X : X \simeq tT(X)$ and $G_Y : Y \simeq Tt(Y)$ given on underlying sets by $x \mapsto \overline{\{x\}}$ and $y \mapsto T(\{y\})$ respectively. (Exercise 1 will be helpful for getting the sheaf maps). Be sure to check that $T(F_X) = G_{T(X)}$ and $t(G_Y) = F_{t(Y)}$, so F and G are truly 'inverses'. Keep in mind that in both categories, maps are completely determined by their effect on the underlying topological spaces, so this makes some compatibility checks painless.

5. It is a very important fact that nearly all geometric facts about reduced schemes locally of finite type over k can be formulated completely within the framework of abstract algebraic sets. In terms of commutative algebra, this comes down to the fact that k-algebras of finite type are Jacobson rings and have all residue fields at maximals equal to k. For example, check:

(i) If $f : X \to Y$ is a morphism of abstract algebraic sets over k, then f is a closed immersion (i.e., f is topologically a closed embedding and $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is surjective) if and only if $t(f)$ is a closed immersion of schemes. Likewise for open immersions.

(ii) Let Y be any affine abstract algebraic set, so $\mathscr{O}_{t(Y)}(t(Y)) = \mathscr{O}_Y(Y)$ is a finite type reduced k-algebra. Using the universal mapping property of affine schemes, there is a natural map of k-schemes $t(Y) \to \mathrm{Spec}(\mathscr{O}_Y(Y))$. Show this is an isomorphism (this is the precise isomorphism which is always implicit when passing between affine abstract algebraic sets and k-schemes, but no one bothers to mention it).

(iii) Let $\mathbf{A}^n_k$, $\mathbf{P}^n_k$ denote the usual abstract algebraic sets over k ($\mathrm{MaxSpec}(k[X_1, \ldots, X_n])$ for the former, the latter defined by gluing affine spaces along open subsets in the usual way). Show that there is a unique isomorphism of k-schemes

$$t(\mathbf{P}^n_k) \simeq \mathrm{Proj}(k[T_0, \ldots, T_n])$$

which is compatible with the $n+1$ standard open immersions of $t(\mathbf{A}^n_k) \simeq \mathrm{Spec}(k[X_1, \ldots, X_n])$ into each side. More generally, let $Y \subseteq \mathbf{P}^n_k$ be a closed subset with homogenous coordinate ring $S(Y)$. Give Y its canonical structure of abstract algebraic set. Show that there is a unique isomorphism of k-schemes $t(Y) \simeq \mathrm{Proj}(S(Y))$ compatible with the natural maps from each to $\mathrm{Proj}(k[T_0, \ldots, T_n])$.

(iv) Let X be a reduced scheme locally of finite type over k. Show that $\dim X = \dim t(X)$ and X is irreducible if and only if $t(X)$ is irreducible. In fact, show that the functor t on topological spaces sets up a bijection between irreducible closed sets in X and $t(X)$. Finally, check that X is quasi-compact if and only if $t(X)$ is.

Much of the above carries over to an arbitrary field k, and with the reducedness hypothesis removed, but it is theoretically clumsy. **From now on, we work exclusively with schemes**. But keep the classical picture in mind; it is very very important.