1. (i) Read the handout on root system decomposition (including a strong uniqueness).
   (ii) Exer. 4 (allowing any char. 0 field \( k \)) and Exer. 6 (over \( \mathbb{Q}, \mathbb{R} \)) in 5.14 of Ch. V. For irreducible \((V, \Phi)\), deduce the \( W(\Phi) \)-action on \( V_k' \) is irreducible for any field extension \( k'/k \).

2. Read the handout computing the root systems for classical groups (types A, B, C, D).
   (i) Usage of the root notation of that handout (not the course text!), for Propositions 6.4 (for \( n \geq 3 \)) and 6.6 in Ch. V verify that the indicated basis in each “part (iii)” is a basis, with associated positive system of roots as in each “part (ii)” and associated Weyl chamber whose closure is indicated in each “part (v)”.
   (ii) Verify the diagram in “part (iv)” for each, labeling nodes by corresponding basis roots.

3. (i) Let \( f : E \to M \) be a covering space of a \( C^\infty \) manifold. Prove \( E \) admits a unique \( C^\infty \) structure making \( f \) a local diffeomorphism. (Prove general uniqueness, then existence for split covers, and then general existence by gluing locally over \( M \) via uniqueness on overlaps.)
   (ii) Let \( M \) be a topological manifold, \( m_0 \in M \), and \( f : E \to M \) a covering space. For \( [\gamma] \in \pi_1(M, m_0) \) and \( x \in E_{m_0} \), define \( x.\gamma := \tilde{\gamma}_x(1) \in E_{m_0} \) where \( \tilde{\gamma}_x : [0, 1] \to E \) is the unique lift of \( \gamma : [0, 1] \to M \) with \( \tilde{\gamma}_x(0) = x \). Prove \( x.\gamma \) depends on \( \gamma \) through \( [\gamma] \), and is a right action of \( \pi_1(M, m_0) \) on \( E_{m_0} \), transitive if \( E \) is connected. Deduc \( f \) splits if \( \pi_1(M, m_0) = 1 \).
   (iii) Any connected topological manifold \( M \) has a connected covering space \( \tilde{M} \to M \) with \( \pi_1(\tilde{M}) = 1 \) (e.g., \( S^1 = \mathbb{R} \)). Prove universality: for covering spaces \( E \to X \), continuous \( h : (M, m_0) \to (X, x_0) \), and \( e_0 \in E_{x_0} \), \( \tilde{m}_0 \in \tilde{M}_{m_0} \), there is a unique continuous \( \tilde{h} : (\tilde{M}, \tilde{m}_0) \to (E, e_0) \) over \( h \). (By (ii), \( \tilde{M} \times_X E \to \tilde{M} \) splits; look at \( \text{pr}_2 \) on the component of \( (\tilde{m}_0, e_0) \).)
   (iv) For a connected Lie group \((G, e)\), show \((\tilde{G}, \tilde{e})\) has a unique Lie group structure making \( \tilde{G} \to G \) a Lie group homomorphism. (Hint: apply (iii) to \( m : G \times G \to G \) as \( h \), etc.) Show the discrete kernel is naturally isomorphic to \( \pi_1(G, e) \) (so \( \pi_1(G, e) \) is commutative).
   (v) Accept \( \pi_1(M)_{ab} = H_1(M, \mathbb{Z}) \) is finitely generated for connected compact manifolds. For a connected compact Lie group \( G \) with \( \#Z_G < \infty \), prove \( \deg(\tilde{G} \to G) < \infty \), so \( \tilde{G} \) is compact. (Use (iv) above and 4(iv) in HW8 to uniformly bound finite quotients of \( \pi_1(G, e) \).)

4. For compact connected \( G \), let \( H \to G/Z^0_G \) be the compact universal cover (see Exer. 3).
   (i) Show \( h = g/\tilde{\rho}_g \), \([h, h] = h\), and \( \text{ad}_g \) factors through a representation \( h \to \text{gl}(g) \).
   (ii) Prove semisimplicity of finite-dimensional \( \mathbb{R} \)-representations of \( h \) ("unitary" trick, as \( \pi_1(H) = 1 \)). Deduce there is a unique \( g \)-equivariant section \( s : h \to g \). Using \( H \to G \) lifting \( s \), show \( G' \) is closed, \((G')' = G'\), and \( Z^0_G \times G' \to G \) is an isogeny (so \( \#Z_G < \infty \) iff \( G = G'! \)).

5. Let \( G \) be a connected compact Lie group with \( \#Z_G < \infty \), \( T \) a maximal torus, and \( \Phi = \Phi(G, T) \). Let \( V = X(T)_{\mathbb{Q}} \), and \( \{(V_i, \Phi_i)\} \) be the set of irreducible components of \((V, \Phi)\).

Read the handout “Simple factors” that uses Exercises 1 and 4 to relate the \( \Phi_i \)'s to the minimal nontrivial connected closed normal subgroups \( G_i \) of \( G \), and to explain how the study of such \( G \) typically reduces to the case of those \( G \) with an irreducible root system (or at least read the statement of Theorem 1.1 there).

Together with the end of Exercise 4(ii), this underlies the precise sense in which the connected Dynkin diagrams encode the building blocks for all connected compact Lie groups.