Math 210C. Homework 8

1. Let \{H_i\} be a locally finite set of affine hyperplanes in \(V = \mathbb{R}^n\), so \(H_i = \ell_i^{-1}(c_i)\) for \(c_i \in \mathbb{R}\) and \(\ell_i \in V^* - \{0\}\). Its chambers are connected components of \(V - (\cup H_i)\). Pick a chamber \(K, x_0 \in K\), and let \(\varepsilon_i = \text{sgn}(\ell_i(x_0) - c_i) = \pm 1\), so \(x_0 \in H_i^{>0} := \{\varepsilon_i(\ell_i - c_i) > 0\}\).

   (i) Prove \(K = \bigcap_i H_i^{>0}\) (convex) and such intersection with any signs \(\varepsilon_i\) is a chamber if \(\neq 0\). Use closed segments to show \(\overline{K} = \bigcap_i H_i^{\geq 0}\) (evident definition for \(H_i^{\geq 0}\)). Deduce \(\text{int}_V \overline{K} = K\). (Hint: For \(x \in \partial_V \overline{K}\), arrange \(x \in \cap H_j\). Find a closed ray \(L\) based at \(x_0 \in K\) missing all \(H_j \cap H_{j'}\) when \(j' \neq j\) but meeting some \(H_j\); study \(L \cap \overline{K}\).) Deduce \(K = \bigcap_{H \in \text{walls}(K)} H^{>0}\) and \(\overline{K} = \bigcap_{H \in \text{walls}(K)} H^{\geq 0}\).

2. (i) Read the handout on dual root systems.

   (ii) Let \(G\) be a connected compact Lie group with \(\#Z_G < \infty\) and choose a maximal torus \(T\), so \((X(T), \Phi)\) is a root system. Show \([X_*(T) : \mathbb{Z}\Phi^\vee] < \infty\). Deduce that for \(G_a := Z_G(T_a)\)' (\(\text{SU}(2)\) or \(\text{SO}(3)\)), we have \(\sum_{\alpha \in \Phi} \text{Lie}(G_a) = g\) (the key is \(t\) is contained in this span).

   (iii) For \(G\) in (ii), prove \(G' := [G, G]\) is a neighborhood of \(e\). Deduce \(G = G'\).

3. Let \((V, \Phi)\) be a root system over a field \(k\) of characteristic 0, and define the canonical symmetric bilinear form \((v | v') = \sum_{\alpha \in \Phi} (v, \alpha') (v', \alpha')\) on \(V\). Read dual root system handout.

   (i) Prove \((\cdot | \cdot)\) is \(W(\Phi)\)-invariant, and moreover positive-definite if \(k = \mathbb{Q}\).

   (ii) Let \(V_0 = Q\Phi\) (so \(k \otimes_{\mathbb{Q}} V_0 = V\); see the handout on dual root systems). Show \((\cdot | \cdot)\) is the scalar extension of the analogue \((\cdot | \cdot)_0\) on \(V_0\), so \(V\) admits a (canonical) non-degenerate \(W(\Phi)\)-invariant symmetric bilinear form (so \((V^*, \Phi^\vee) \simeq (V, \Phi)\) via \(a^\vee \mapsto a' := 2a/(a|a))\).

   (iii) For \(\Psi \subset \Phi\) and \(V' = k\Psi\), show \((V', \Phi \cap \mathbb{Z}\Psi)\) is a root system with coroots \(a^\vee |_{V'}\). Also show \(V^{W(\Phi)} = 0\) (hint: consider \(W(\Phi)\)-equivariant quotients of \(V\) with trivial \(W(\Phi)\)-action).

4. Let \(G\) be a connected compact Lie group, and \(T\) a maximal torus of \(G\); we make no finiteness hypotheses on \(Z_G\). Define \(V = X(T)_Q\).

   (i) For \(a \in \Phi\) let \(a^\vee \in X_*(T) \subset V^*\) be the coroot arising from \((Z_G(T_a)', T_a, a|T_a)\) as in the handout on coroots. Show \(X_*(T)Q = X_*(T/Z_G)Q = X(T/Z_G)^\ast\) carries \(a^\vee\) to the coroot associated to \(a\) for \((G/Z_G, T/Z_G)\), and for \(b \in \Phi\) prove \(b^\vee = a^\vee\) if and only if \(b = a\).

   (ii) Let \(\Phi^\vee\) be the set of coroots for \((G, T)\). Prove \(Q\Phi^\vee\) is a complement to \(X_*(Z_G^0)Q = X_*(Z_G)Q\) in \(X_*(T)Q\) (equivalently, it maps isomorphically onto \(X_*(T/Z_G)Q = X(T/Z_G)^\ast\)) inducing an isomorphism \(W(G, T) \simeq W(\Phi(G/Z_G, T/Z_G)) = W(\Phi(G, T))\). Hint: make a \(W(G, T)\)-invariant positive-definite quadratic form \(q : X(T)_Q \to Q\).

   (iii) In (ii) you showed \(X_*(Z_G)Q\) has a canonical \(W(G, T)\)-equivariant complement in \(X_*(T)Q\) (so likewise for \((Z_G^0)_Q\) inside \(X(T)_Q\)).

   (iv) Assume \(Z_G\) is finite. For any lattice \(L\) in \(V^*\) (e.g., \(Z\Phi^\vee\)), let \(L'\) denote the \(Z\)-dual Hom\((L, Z) \subset \text{Hom}(L, Z)_Q = V^{**} = V\) consisting of \(v \in V\) such that \(\ell(v) \in Z\) for all \(\ell \in L\). For any isogeny \(\pi : \mathcal{G} \to G\) from a connected compact Lie group \(\mathcal{G}\), show its maximal torus \(\mathcal{T} = \pi^{-1}(T)\) satisfies \(Z\Phi^\vee \subset X(T) \subset X(\mathcal{T}) \subset (Z\Phi^\vee)'\) inside \(V\) and prove \(#\ker \pi = [X(\mathcal{T}) : X(T)]\). Deduce an upper bound on \# \ker \pi determined entirely by \((V, \Phi)\).

   (This will underlie our later proof that \(\pi_1(G)\) is finite when \(Z_G\) is finite.)