Math 210C. Homework 6

1. (i) Let $G$ be a unimodular Lie group, and $K$ and $B$ closed subgroups such that $K$ is unimodular and multiplication $K \times B \to G$ is a diffeomorphism. (HW1 and HW2 give some such $G$ for which unimodularity admits a unified proof via algebraic-group techniques.) Let $dk$ be a Haar measure on $K$ and $db$ be a right Haar measure on $B$. Let $dg$ be a Haar measure on $G$. Prove the product measure $dk \cdot db$ is equal to $f(g)dg$ for some $f \in C^\infty(G)$, and then use invariance properties of the measures to deduce $f$ is constant!

(ii) Let $U \subset G := \text{SL}_n(\mathbb{R})$ be the subgroup of upper-triangular unipotents, $A \subset G$ the “positive” diagonal subgroup, and $K = \text{SO}(n)$. Prove $du := \prod_{i<j} du_{ij}$ is a right Haar measure on $A$, and $dk \cdot du$ is a Haar measure on $G$ for a Haar measure $d\mu$ on $K$. (Proving invariance of $dk \cdot da \cdot du$ by bare hands is a mess!)

(iii) Make (ii) explicit for $\text{SL}_2(\mathbb{R})$ via the equality $SO(2) = S^1$.

2. By IV, 3.13 (whose proof you should read!), for $G = U(n)$ and the diagonal maximal torus $T = (S^1)^n$, the injection $R(G) \hookrightarrow R(T)^W$ is an equality. The idea is to compute $R(T)^W$ as a ring and show that characters of specific $G$-representations (in fact, the first $n$ exterior powers of the standard representation of $G = U(n)$ and the dual of the $n$th) generate $R(T)^W$.

Explicitly, $R(T) = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] = \mathbb{Z}[z_1, \ldots, z_n, 1/\prod z_i]$ on which $W = S_n$ acts by usual permutations, so $R(T)^W = \mathbb{Z}[z_1, \ldots, z_n]^W[1/\prod z_i]$. Letting $\sigma_j = \sigma_j(z_1, \ldots, z_n)$ be the $j$th symmetric polynomial $(1 \leq j \leq n)$, so $\prod z_i = \sigma_n$, (i) below computes $R(T)^W$.

(i) Using basic properties of transcendence degree, sketch the Galois-theoretic proof that $\sigma_1, \ldots, \sigma_n$ are algebraically independent in $k[z_1, \ldots, z_n]$ and $k(\sigma_1, \ldots, \sigma_n) = k(z_1, \ldots, z_n)^{S_n}$ for any field $k$. Using integrality considerations, deduce $k[z_1, \ldots, z_n]^{S_n} = k[\sigma_1, \ldots, \sigma_n]$.

(ii) Prove $A[z_1, \ldots, z_n]^{S_n} = A[\sigma_1, \ldots, \sigma_n] = A[\sigma_1, \ldots, \sigma_n]$ for $A = \mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$ ($m > 0$).

(iii) For any commutative ring $A$, prove $\sigma_1, \ldots, \sigma_n$ are algebraically independent over $A$ inside $A[z_1, \ldots, z_n]$ and that $A[\sigma_1, \ldots, \sigma_n] = A[z_1, \ldots, z_n]^{S_n}$.

3. IV, 3.14, Exercise 3.


(ii) Using $R(\text{SU}(2)) \simeq \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_2 - 1) = \mathbb{Z}[\sigma_1]$, describe the character $\chi_m$ of $V_m$ as a polynomial in $\sigma_1$ for $m \leq 5$. (For example, $t^2 + 1 + t^{-2} = (t + t^{-1})^2 - 1$, so $\chi_2 = \sigma_1^2 - 1$.)

5. For $\mathbb{C}$-valued continuous class functions $f$ on $\text{SU}(2)$ and the volume-1 measure $dg$, prove

$$\int_{\text{SU}(2)} f(g)dg = 2 \int_0^1 f(t(\theta)) \sin^2(2\pi\theta) d\theta$$

where $t(\theta) = \text{diag}(e^{2\pi i \theta}, e^{-2\pi i \theta})$. (This agrees with II, 5.2 via change of variable. As a check on the normalizations, you can handle $f = 1$ by hand.) Also find an analogue for $\text{SO}(3)$.

6. Let the group $\text{SU}(2)$ of norm-1 quaternions act on $H = \mathbb{R}^4$ in two commuting ways: left-multiplication ($v \mapsto vu$ for $v \in H$) and right-multiplication through inversion ($v \mapsto vu^{-1}$ for $v \in H$). These actions define a Lie group homomorphism $f : \text{SU}(2) \times \text{SU}(2) \to \text{GL}_4(\mathbb{R})$.

(i) Check $\ker f$ is the diagonally embedded $\mu_2 = \{ \pm 1 \}$, and that $f$ lands inside $\text{SO}(4)$.

(ii) Prove that the map $(\text{SU}(2) \times \text{SU}(2))/\mu_2 \to \text{SO}(4)$ is a Lie group isomorphism.