Math 210C. Representations of $\mathfrak{sl}_2$

1. Introduction

In this handout, we work out the finite-dimensional k-linear representation theory of $\mathfrak{sl}_2(k)$ for any field k of characteristic 0. (There are also infinite-dimensional irreducible k-linear representations, but here we focus on the finite-dimensional case.) This is an introduction to ideas that are relevant in the general classification of finite-dimensional representations of "semisimple" Lie algebras over fields of characteristic 0, and is a *crucial* technical tool for our later work on the structure of general connected compact Lie groups (especially to explain the ubiquitous role of SU(2) in the general structure theory).

When k is fixed during a discussion, we write $\mathfrak{sl}_2$ to denote the Lie algebra $\mathfrak{sl}_2(k)$ of traceless 2×2 matrices over k (equipped with its usual Lie algebra structure via the commutator inside the associative k-algebra $\mathrm{Mat}_2(k)$). Recall the standard k-basis $\{X^-, H, X^+\}$ of $\mathfrak{sl}_2$ given by

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

satisfying the commutation relations

$$[H, X^{\pm}] = \pm 2X^{\pm},\ \ [X^+, X^-] = H.$$

For an $\mathfrak{sl}_2$-module V over k (i.e., k-vector space V equipped with a map of Lie algebras $\mathfrak{sl}_2 \to \mathrm{End}_k(V)$), we say it is *irreducible* if $V \neq 0$ and there is no nonzero proper $\mathfrak{sl}_2$-submodule. We say that V is *absolutely irreducible* (over k) if for any extension field K/k the scalar extension V_K is irreducible as a K-linear representation of the Lie algebra $\mathfrak{sl}_2(K) = K \otimes_k \mathfrak{sl}_2$ over K.

If V is a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ over k then we define the *dual* representation to be the dual space V^* equipped with the $\mathfrak{g}$-module structure $X.\ell = \ell\circ(-X.v) = -\ell(X.v)$ for $\ell \in V^*$. (The reason for the minus sign is to ensure that the action of $[X, Y]$ on V^* satisfies $[X, Y](\ell) = X.(Y.\ell) - Y.(X.\ell)$ rather than $[X, Y](\ell) = Y.(X.\ell) - X.(Y.\ell)$. The intervention of negation here is similar to the fact that dual represenations of groups G involve evaluation against the action through inversion, to ensure the dual of a left G-action is a left G-action rather than a right G-action.) The natural k-linear isomorphism $V \simeq V^{**}$ is easily checked to be $\mathfrak{g}$-linear. It is also straightforward to check (do it!) that if $\rho : G \to \mathrm{GL}(V)$ is a smooth representation of a Lie group G on a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$ then $\mathrm{Lie}(\rho^*) = \mathrm{Lie}(\rho)^*$ as representations of $\mathrm{Lie}(G)$ (where V^* is initially made into a G-representation in the usual way). The same holds if G is a complex Lie group (acting linearly on finite-dimensional $\mathbf{C}$-vector space via a holomorphic ρ, using the $\mathbf{C}$-linear identification of $\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}_{\mathbf{C}}(V)$ for $\mathrm{GL}(V)$ as a complex Lie group).

In the same spirit, the *tensor product* of two $\mathfrak{g}$-modules V and V' has underlying k-vector space $V \otimes_k V'$ and $\mathfrak{g}$-action given by

$$X.(v \otimes v') = (X.v) \otimes v' + v \otimes (X.v')$$

for $X \in \mathfrak{g}$ and $v \in V$, $v' \in V'$. It is easy to check (do it!) that this tensor product construction is compatible via the Lie functor with tensor products of finite-dimensional representations of Lie groups in the same sense as formulated above for dual representations.

2. Primitive vectors

To get started, we prove a basic fact:

Lemma 2.1. *Let V be a nonzero finite-dimensional $\mathfrak{sl}_2$-module over k. The operators $X^{\pm}$ on V are nilpotent and the H-action on V carries each $\ker(X^{\pm})$ into itself.*

Proof. Since $[H, X^{\pm}] = \pm 2X^{\pm}$, the Lie algebra representation conditions give the identity

$$H.(X^{\pm}.v) = [H, X^{\pm}].v + X^{\pm}.(H.v) = \pm 2X^{\pm}.v + X^{\pm}.(H.v)$$

for any $v \in V$. Thus, if $X^{\pm}v = 0$ then $X^{\pm}.(H.v) = 0$, so $H.v \in \ker X^{\pm}$. This gives the H-stability of each $\ker X^{\pm}$.

We now show that $X^{\pm}$ is nilpotent. More generally, if E and H are linear endomorphisms of a finite-dimensional vector space V in characteristic 0 and the usual commutator $[E, H]$ of endomorphisms is equal to cE for some nonzero c then we claim that E must act nilpotently on V. By replacing H with $(1/c)H$ if necessary, we may assume $[E, H] = E$. Since $EH = HE + E = (H + 1)E$, we have

$$E^2H = E(H + 1)E = (EH + E)E = ((H + 1)E + E)E = (H + 2)E^2,$$

and in general $E^nH = (H + n)E^n$ for integers $n \geq 0$. Taking the trace of both sides, the invariance of trace under swapping the order of multiplication of two endomorphisms yields

$$\mathrm{Tr}(E^nH) = \mathrm{Tr}((H + n).E^n) = \mathrm{Tr}(E^n.(H + n)) = \mathrm{Tr}(E^nH) + n\mathrm{Tr}(E^n),$$

so $n\mathrm{Tr}(E^n) = 0$ for all $n > 0$. Since we're in characteristic 0, we have $\mathrm{Tr}(E^n) = 0$ for all $n > 0$.

There are now two ways to proceed. First, we can use "Newton's identities", which reconstruct the symmetric functions of a collection of d elements $\lambda_1, \dots, \lambda_d$ (with multiplicity) in a field of characteristic 0 (or any commutative $\mathbf{Q}$-algebra whatsoever) from their first d power sums $\sum_j \lambda_j^n$ $(1 \leq n \leq d)$. The formula involves division by positive integers, so the characteristic 0 hypothesis is essential (and the assertion is clearly false without that condition). In particular, if the first d power sums vanish then all λ_j vanish. Applying this to the eigenvalues of E over $\overline{k}$, it follows that the eigenvalues all vanish, so E is nilpotent. However, the only proofs of Newton's identities that I've seen are a bit unpleasant (perhaps one of you can enlighten me as to a slick proof?), so we'll instead use a coarser identity that is easy to prove.

In characteristic 0 there is a general identity (used very creatively by Weil in his original paper on the Weil conjectures) that reconstructs the "reciprocal root" variant of characteristic polynomial of any endomorphism E from the traces of its powers: since the polynomial $\det(1 - tE)$ in $k[t]$ has constant term 1 and $\log(1 - tE) = -\sum_{n\geq 1}(tE)^n/n \in t\mathrm{Mat}_d(k[\![t]\!])$, it makes sense to compute the trace $\mathrm{Tr}(\log(1 - tE)) \in tk[\![t]\!]$ and we claim that

$$\begin{aligned} \det(1 - tE) = \exp(\log(\det(1 - tA)) &= \exp(\mathrm{Tr}(\log(1 - tA))) \\ &= \exp(\mathrm{Tr}(-\sum_{n\geq 1} t^nE^n/n)) \\ &= \exp(-\sum_{n\geq 1} t^n\mathrm{Tr}(E^n)/n) \end{aligned}$$

as formal power series in $k[\![t]\!]$. (Note that it makes sense to plug an element of $tk[\![t]\!]$ into any formal power series in one variable!) Once this identity proved for general E, it follows that if $\mathrm{Tr}(E^n)$ vanishes for all $n > 0$ then $\det(1 - tE) = \exp(0) = 1$. But the coefficients of the positive powers of t in $\det(1 - tE)$ are the lower-order coefficients of the characteristic polynomial of E, whence this characteristic polynomial is t^d, so E is indeed nilpotent, as desired.

In the above string of equalities, the only step which requires explanation is the equality

$$\log(\det(1 - tE)) = \mathrm{Tr}(\log(1 - tE))$$

in $k[\![t]\!]$. To verify this identity we may extend scalars so that k is algebraically closed, and then make a change of basis on k^d so that E is upper-triangular, say with entries $\lambda_1, \dots, \lambda_d \in k$ down the diagonal. Thus, $\det(1 - tE) = \prod(1 - \lambda_j t)$ and $\log(1 - tE) = -\sum_{n \geq 1} t^n E^n/n$ is upper-triangular, so its trace only depends on the diagonal, whose entries are $\log(1 - \lambda_j t) \in tk[\![t]\!]$. Summarizing, our problem reduces to the formal power series identity that log applied to a finite product of elements in $1 + tk[\![t]\!]$ is equal to the sum of the logarithms of the terms in the product. By continuity considerations in complete local noetherian rings (think about it!), the rigorous justification of this identity reduces to the equality $\log(\prod(1 + x_j)) = \sum \log(1 + x_j)$ in $\mathbf{Q}[\![x_1, \dots, x_d]\!]$, which is easily verified by computing coefficients of multivariable Taylor expansions. ■

In view of the preceding lemma, if V is any nonzero finite-dimensional $\mathfrak{sl}_2$-module over k we may find nonzero elements $v_0 \in \ker X^+$, and moreover if k is algebraically closed then we can find such v_0 that are eigenvectors for the restriction of H to an endomorphism of $\ker X^+$.

Definition 2.2. A *primitive vector* in V is an H-eigenvector in $\ker X^+$.

We shall see that in the finite-dimensional case, the H-eigenvalue on a primitive vector is necessarily a non-negative integer. (For infinite-dimensional irreducible $\mathfrak{sl}_2$-modules one can make examples in which there are primitive vectors but their H-eigenvalue is not an integer.) We call the H-eigenvalue on a primitive eigenvector (or on any H-eigenvector at all) its *H-weight*.

3. Structure of $\mathfrak{sl}_2$-modules

Here is the main result, from which everything else will follow.

Theorem 3.1. *Let $V \neq 0$ be a finite-dimensional $\mathfrak{sl}_2$-module over a field k with* $\mathrm{char}(k) = 0$.

(1) *The H-weight of any primitive vector is a non-negative integer.*
(2) *Let $v_0 \in V$ be a primitive vector, its weight an integer $m \geq 0$. The $\mathfrak{sl}_2$-submodule $V' := \mathfrak{sl}_2 . v_0$ of V generated by v_0 is absolutely irreducible over k and has dimension $m + 1$. Moreover, if we define*

$$v_j = \frac{1}{j!}(X^-)^j(v_0)$$

for $0 \leq j \leq m$ (and define $v_{-1} = 0$, $v_{m+1} = 0$) then

$$H.v_j = (m - 2j).v_j, \quad X^+.v_j = (m - j + 1)v_{j-1}, \quad X^-.v_j = (j + 1)v_{j+1}$$

for $0 \leq j \leq m$*. In particular, the* H*-action on* V' *is diagonalizable with eigenspaces of dimension* 1 *having as eigenvalues the* $m+1$ *integers* $\{m, m-2, \ldots, -m+2, -m\}$*.*

(3) $X^+|_{V'}$ *has kernel equal to* kv_0*, and this line exhibits the unique highest* H*-weight.*

In particular, if V *is irreducible then* $V = V'$ *is absolutely irreducible and is determined up to isomorphism by its dimension* $m+1$*, and all* H*-eigenvalues on* V' *are integers, with the unique highest weight* m *having a* 1*-dimenional eigenspace.*

Before proving the theorem, we make some remarks.

Remark 3.2. In class we saw the visualization of the effect of $X^{\pm}$ on the H-eigenlines, as "raising/lowering" operators with respect to the H-eigenvalues (hence the notation X^+ and X^-, the asymmetry between which is due to our convention to work with $H = [X^+, X^-]$ rather than $-H = [X^-, X^+]$ at the outset).

The conceptual reason that we divide by factorials in the definition of the v_j's is to ensure that the formulas relating $X^{\pm}.v_j$ to $v_{j\mp 1}$ involve *integer* coefficients with the evident monotonicity behavior as we vary j. In view of the fact that we'll later *construct* such an irreducible $(m+1)$-dimensional representation as the mth symmetric power of the *dual* of the standard 2-dimensional representation of $\mathfrak{sl}_2$, what is really going on with the factorial division is that the formation of symmetric powers of finite-dimensional vector spaces does *not* naturally commute with the formation of dual spaces (in contrast with tensor powers and exterior powers): in positive characteristic it fails badly, and in general the symmetric power of a finite-dimensional dual vector space is identified with the dual of a "symmetric divided power" space (and divided powers are identified with symmetric powers in characteristic 0 via suitable factorial divisions); read the Wikipedia page on *divided power structure*.

Remark 3.3. The final part of Theorem 3.1, characterizing an irreducible $\mathfrak{sl}_2$-module up to isomorphism by its highest weight, has a generalization to all "semisimple" finite-dimensional Lie algebras over algebraically closed fields of characteristic 0, called the *Theorem of the Highest Weight*. The precise statement of this result requires refined knowledge of the structure theory of semisimple Lie algebras (such as results and definitions concerning Cartan subalgebras), so we do not address it here.

In addition to the consequence that any irreducible $\mathfrak{sl}_2$-module of finite dimension over k is absolutely irreducible and determined up to isomorphism by its dimension (so it is isomorphic to its dual representation!), there is the separate problem of showing that all possible dimensions really occur. That is, one has to make an actual construction of an irreducible $\mathfrak{sl}_2$-module of every positive dimension. Likewise, the precise statement of the Theorem of the Highest Weight for general semisimple finite-dimensional Lie algebras includes an existence aspect, and is very much tied up with a good knowledge of the theory of root systems (a theory that plays an essential role in our later work on the structure theory of connected compact Lie groups).

We will address the existence issue below for $\mathfrak{sl}_2$, and also show that the entire finite-dimensional representation theory of $\mathfrak{sl}_2$ is completely reducible – i.e., every such representation is a direct sum of irreducibles – a fact that we can see over $k = \mathbf{C}$ via using the analytic technique of Weyl's unitarian trick to pass to an analogue for the connected compact Lie group SU(2). The proof of such complete reducibility over a general field of characteristic

0 (especially not algebraically closed) requires a purely algebraic argument. In particular, this will give a purely algebraic proof of the semisimplicity of the finite-dimensional **C**-linear representation theory of $\mathfrak{sl}_2(\mathbf{C})$ without requiring the unitarian trick (even though historically the unitarian trick was a milestone in the initial development of the understanding of the representation theory of finite-dimensional semisimple Lie algebras over **C**).

Now we finally prove Theorem 3.1.

Proof. In view of the precise statement of the theorem, it is sufficient to prove the result after a ground field extension (check: for this it is essential that we are claiming that certain eigenvalues are integers, not random elements of k). Thus, now we may and do assume k is algebraically closed, so we can make eigenvalues. Consider any $\lambda \in k$ and $v \in V$ satisfying $H.v = \lambda v$. (We are mainly interested in the case $v \neq 0$, but let's not assume that just yet). The condition of V being an $\mathfrak{sl}_2$-module yields the computation

$$H.(X^{\pm}.v) = [H, X^{\pm}].v + X^{\pm}.H.v = \pm 2X^{\pm}.v + \lambda X^{\pm}.v = (\lambda \pm 2)X^{\pm}.v$$

that we saw earlier. In particular, if v is a λ-eigenvector for H then $X^{\pm}.v$ is an eigenvector for H with eigenvalue $\lambda \pm 2j$ provided that $X^{\pm}.v \neq 0$. In particular, the elements in the sequence $\{(X^{\pm})^j.v\}_{j\geq 0}$ that are nonzero are mutually linearly independent since they are H-eigenvectors with pairwise distinct eigenvalues $\lambda \pm 2j$. (Here we use that we're in characteristic 0!)

Let v_0 be a primitive vector, which exists by Lemma 2.1 since k is algebraically closed. Thus, $H.v_0 = \lambda v_0$ for some $\lambda \in k$. Define $v_j = (1/j!)(X^-)^j.v_0$ for all $j \geq 0$, and define $v_{-1} = 0$. Since X^- is nilpotent on V, we have $v_j = 0$ for sufficiently large $j > 0$, so the set of j such that $v_j \neq 0$ is a sequence of consecutive integers $\{0, 1, \dots, m\}$ for some $m \geq 0$. Clearly from the definition we have

$$H.v_j = (\lambda - 2j)v_j, \;\; X^-.v_j = (j+1)v_{j+1}$$

for $j \geq 0$. We claim that $X^+.v_j = (\lambda - j + 1)v_{j-1}$ for all $j \geq 0$. This is clear for $j = 0$, and in general we proceed by induction on j. Assuming $j > 0$ and that the result is known for $j - 1 \geq 0$, we have

$$X^+.v_j = (1/j)X^+.X^-.v_{j-1} = (1/j)([X^+, X^-] + X^-.X^+).v_{j-1} = (\lambda - j + 1)v_{j-1},$$

where the final equality is a computation using the inductive hypothesis that we leave to the reader to check.

We know that $v_0, \dots, v_m$ are linearly independent. Since

$$(\lambda - m)v_m = X^+.v_{m+1} = 0,$$

necessarily $\lambda = m$. This proves that the primitive vector v_0 has H-weight equal to the non-negative integer m, and from our formulas for the effect of H and $X^{\pm}$ on each v_j ($0 \leq j \leq m$), clearly the $(m+1)$-dimensional k-linear span V' of $v_0, \dots, v_m$ coincides with the $\mathfrak{sl}_2$-submodule of V generated by v_0. The formulas show that $X^+|_{V'}$ has kernel equal to the line spanned by v_0.

It remains to show that V' is irreducible as an $\mathfrak{sl}_2$-module. (By extending to an even larger algebraically closed extension if necessary and applying the same conclusion over that field, the absolute irreducibility would follow.) Consider a nonzero $\mathfrak{sl}_2$-submodule W of V'.

Since the H-action on V is diagonalizable with 1-dimensional eigenspaces, the H-stable W must be a span of some of these eigenlines. But the explicit formulas for the effect of $X^{\pm}$ as "raising/lowering" operators on the lines kv_j makes it clear that a single such line generates the entirety of V' as an $\mathfrak{sl}_2$-module. Hence, V' is irreducible. ■

4. Complete reduciblity and existence theorem

We finish by discussing two refinements: the proof that every finite-dimensional $\mathfrak{sl}_2$-module is a direct sum of irreducibles, and the existence of irreducible representations of each positive dimension. As a consequence, for $k = \mathbf{C}$ we'll recover the connection between irreducible finite-dimensional SO(3)-representations over $\mathbf{C}$ and spherical harmonics. First we prove the existence result.

Proposition 4.1. *Let V_1 be the standard 2-dimensional representation of $\mathfrak{sl}_2 \subset \mathrm{End}_k(k^2)$. For $m \geq 0$, the symmetric power $V_m = \mathrm{Sym}^m(V_1^*)$ of dimension $m+1$ is irreducible as an $\mathfrak{sl}_2$-module for every $m \geq 1$.*

Proof. Obviously $V_0 = k$ is the 1-dimensional trivial representation, so we may focus on cases with $m \geq 1$. In an evident manner, V_m is the space of homogenous polynomials of degree m in two variables z_1, z_2. By inspection, $v_0 := z_1^m$ is a primitive vector, and the associated v_j's are given by

$$v_j = \binom{m+1}{j} z_1^{m-j} z_2^j$$

for $0 \leq j \leq m$. These span the entire $(m+1)$-dimensional space V_m, so V_m is irreducible by Theorem 3.1. ■

In the general theory of semisimple Lie algebras, there is a construction called the *Killing form* (named after Wilhelm Killing even though it was introduced by Cartan, much as Cartan matrices were introduced by Killing...). This underlies the conceptual technique by which complete reducibility of representations is proved. In our situation we will use our explicit knowledge of the list of irreducibles to prove the complete reducibility; such a technique is certainly ill-advised in a broader setting (beyond $\mathfrak{sl}_2$):

Theorem 4.2. *Every nonzero finite-dimensional $\mathfrak{sl}_2$-module over k is a direct sum of irreducibles.*

Proof. We proceed by induction on the dimension, the case of dimension 1 being clear. Consider a general V, so if it is irreducible then there is nothing to do. Hence, we may assume it contains a nonzero proper $\mathfrak{sl}_2$-submodule, so there is a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of $\mathfrak{sl}_2$-modules, with V' and V'' nonzero of strictly smaller dimension than V. Hence, V' and V'' are each a direct sum of irreducibles. We just want to split this exact sequence of $\mathfrak{sl}_2$-modules.

As we noted in class, for any Lie algebra $\mathfrak{g}$ over k, the category of $\mathfrak{g}$-modules over k is the same as the category of left $U(\mathfrak{g})$-modules where $U(\mathfrak{g})$ is the associative universal enveloping algebra over k. This is the quotient of the tensor algebra $\oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$ modulo the 2-sided ideal generated by relations $X \otimes Y - Y \otimes X = [X, Y]$ for $X, Y \in \mathfrak{g}$. (The Poincaré–Birkhoff–Witt theorem describes a basis of $U(\mathfrak{g})$, but we don't need that.) Short exact sequences of $\mathfrak{g}$-modules are the same as those of left $U(\mathfrak{g})$-modules. Letting $R = U(\mathfrak{sl}_2)$, the short exact sequence of interest is an element of $\mathrm{Ext}^1_R(V'', V')$. We want this Ext-group to vanish.

In an evident manner, since V'' is finite-dimensional over k, we see that for any left R-module W (even infinite-dimensional), $\mathrm{Hom}_R(V'', W) \simeq \mathrm{Hom}_R(k, V''^* \otimes_k W)$. By a universal δ-functor argument (or more hands-on arguments that we leave to the interested reader),

$$\mathrm{Ext}^\bullet_R(V'', \cdot) = \mathrm{Ext}^\bullet_R(k, V''^* \otimes_k (\cdot)).$$

(Recall our discussion at the outset of this handout concerning duals and tensor products of representations of Lie algebras.) Thus, to prove the vanishing of the left side in degree 1 when evaluated on a finite-dimensional argument, it suffices to prove the vanishing of the right side in such cases. In other words, we are reduced to proving $\mathrm{Ext}^1_R(k, W) = 0$ for all finite-dimensional left $\mathfrak{sl}_2$-modules W. By using short exact sequences in W, we filter down to the case when W is irreducible. Thus, we're reduced to proving the splitting in the special case when $V'' = k$ is the trivial representation and V' is irreducible.

Dualizing is harmless, so we want to split short exact sequences

$$0 \to k \to E \to V_m \to 0$$

for $m \geq 0$. The key trick, inspired by knowledge later in the theory (the structure of the center of the universal enveloping algebra of a semisimple Lie algebra), is to consider the element

$$C := H^2 + 2(X^+X^- + X^-X^+) = H^2 + 2H + 4X^-X^+ \in R = U(\mathfrak{sl}_2).$$

The advanced knowledge that inspires the focus on C is that $C/8$ is a distinguished element in the *center* of $U(\mathfrak{sl}_2)$ (with an analogue in the center of $U(\mathfrak{g})$ for any finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over k), called the *Casimir element*. By centrality it must act as a constant on every absolutely irreducible finite-dimensional representation, due to Schur's Lemma. For $\mathfrak{sl}_2$ one can verify by direct computation (do it!) that C acts as $m(m+2)$ on V_m (be careful about computations with the "boundary" vectors v_0 and v_m in the $(m+1)$-dimensional V_m), so it "picks out" isotypic parts in a direct sum of irreducibles. The centrality of C in $U(\mathfrak{sl}_2)$ will be used in the *proof* of complete reducibility; there is a conceptual proof using the notion of Killing form, but for $\mathfrak{sl}_2$ it's just as easy to give a direct check:

Lemma 4.3. *The element C in $R = U(\mathfrak{sl}_2)$ is central.*

Proof. This amounts to showing that the commutators $CH - HC$ and $CX^\pm - X^\pm C$ in R vanish. By direct computation with commutators in R and the commutator relations in $\mathfrak{sl}_2$, the second expression for C yields

$$[C, H] = 4([X^-, H]X^+ + X^-[X^+, H]) = 4(2X^-X^+ - 2X^-X^+) = 0.$$

By symmetry in the inital expression for C (and its invariance under swaping the roles of X^+ and X^-, which amounts to replacing H with $-H$), to prove that $[C, X^\pm] = 0$ in R it suffices to treat the case of X^+.

Again using the second expression for C, since $[H, X^+] = 2X^+$ we have

$$[C, X^+] = [H^2, X^+] + 4X^+ + 4[X^-X^+, X^+].$$

But $[H^2, X^+] = 4X^+(H+1)$ because

$$\begin{aligned} H^2X^+ - X^+H^2 = H([H, X^+] + X^+H) - X^+H^2 &= H(2X^+ + X^+H) - X^+H^2 \\ &= HX^+(2+H) - X^+H^2 \end{aligned}$$

and substituting the identity $HX^+ = [H, X^+] + X^+H = 2X^+ + X^+H = X^+(2+H)$ yields

$$H^2X^+ - X^+H^2 = X^+(2+H)^2 - X^+H^2 = X^+(4+4H).$$

Thus, $[C, X^+] = 4X^+(H+1) + 4X^+ + 4(X^-(X^+)^2 - X^+X^-X^+)$. Since $X^-X^+ = [X^-, X^+] + X^+X^- = -H + X^+X^-$, inside R we have

$$X^-(X^+)^2 - X^+X^-X^+ = -HX^+ + X^+X^-X^+ - X^+X^-X^+ = -HX^+,$$

and hence

$$CX^+ - X^+C = 4X^+(H+1) + 4X^+ - 4HX^+ = 8X^+ - 4[H, X^+] = 0.$$

■

The upshot of our study of C is that the given short exact sequence is C-equivariant with C acting on V_m via multiplication by $m(m+2)$, and the C-action on E is $\mathfrak{sl}_2$-equivariant since C is *central* in R. Thus, if $m > 0$ (so $m(m+2) \neq 0$ in k) then the $m(m+2)$-eigenspace for C on E is an $\mathfrak{sl}_2$-subrepresentation which does not contain the trivial line k and so must map isomorphically onto the irreducible V_m, thereby splitting the exact sequence as $\mathfrak{sl}_2$-modules. If instead $m = 0$ then the representation of $\mathfrak{sl}_2$ on E corresponds to a Lie algebra homomorphism

$$\mathfrak{sl}_2 \to \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$$

which we want to vanish (so as to get the "triviality" of the $\mathfrak{sl}_2$-action on E, hence the desired splitting). In other words, it suffices to show that $\mathfrak{sl}_2$ does not admit the trivial 1-dimensional Lie algebra as a quotient. Since $[H, X^\pm] = \pm 2X^\pm$, any abelian Lie algebra quotient of $\mathfrak{sl}_2$ must kill $X^\pm$, so it also kills $[X^+, X^-] = H$ and the abelian quotient vanishes. ■

Remark 4.4. In HW4, explicit models are given for the finite-dimensional irreducible $\mathbf{C}$-linear representations of SO(3) via harmonic polynomials in 3 variables (with coefficients in $\mathbf{C}$). These are the representations V_m of SU(2) of dimension $m = 2\ell + 1$. Under the identification of $\mathfrak{so}(3)_{\mathbf{C}}$ with $\mathfrak{sl}_2(\mathbf{C})$, the set of H-eigenvalues is $\{-2\ell, -2\ell+2, \dots, 0, \dots, 2\ell-2, 2\ell\}$, and an explicit nonzero element with H-eigenvalue 0 is computed in pp. 118–121 in the course text. In terms of spherical coordinates (r, θ, φ), such an eigenvector is the ℓth Legendre polynomial evaluated at $\cos\varphi$. (The course text swaps the usual meaning of θ and φ, so it writes $\cos\theta$ for what is usually denoted as $\cos\varphi$.)

When switching between representations of SO(3) and $\mathfrak{sl}_2(\mathbf{C})$ via the unitarian trick, Lie-theoretic invariance under H translates into invariance under the 1-parameter subgroup of SO(3) given by rotation of arbitrary angles around the z-axis (an obvious property for polynomials in $\cos\varphi$!) since the velocity at e for this 1-parameter subgroup turns out to be a $\mathbf{C}^\times$-multiple of H via the isomorphism $\mathfrak{sl}_2(\mathbf{C}) \simeq \mathfrak{so}(3)_{\mathbf{C}}$.