Math 210C. More applications of the exponential map

This handout addresses two applications of the exponential map: continuous homomorphisms between Lie groups are $C^\infty$ and a neat proof of the Fundamental Theorem of Algebra.

1. Continuous homomorphisms

Let $f : G' \to G$ be a continuous homomorphism between Lie groups. We aim to prove that $f$ is necessarily $C^\infty$. Consider the graph map

$$\Gamma_f : G' \to G' \times G$$

defined by $g' \mapsto (g', f(g'))$. This is a continuous injection with inverse given by the continuous restriction of $\text{pr}_1$, so it is a homeomorphism onto its image. Moreover, this image is closed, since it is the preimage of the diagonal in $G \times G$ under the continuous map $G' \times G \to G \times G$ defined by $(g', g) \mapsto (f(g'), g)$. The homomorphism property of $f$ ensures that $\Gamma_f$ is a subgroup inclusion, so $\Gamma_f$ identifies $G'$ topologically with a closed subgroup of $G' \times G$.

As we mentioned in class (and is proved in Theorem 3.11 of Chapter I of the course text, using the exponential map in an essential manner), a closed subgroup of a Lie group is necessarily a closed smooth submanifold. Thus, the graph has a natural smooth manifold structure, and we shall write $H \subseteq G' \times G$ to denote this manifold structure on $\Gamma_f(G')$.

Observe that topologically, $f$ is the composition of the inverse of the homomorphism $\text{pr}_1 : H \to G$ with the other projection $\text{pr}_2 : H \to G$. Since the inclusion of $H$ into $G' \times G$ is $C^\infty$ and $\text{pr}_2 : G' \times G \to G$ is $C^\infty$, it follows that $\text{pr}_2 : H \to G$ is $C^\infty$. Hence, to deduce that $f$ is $C^\infty$ it suffices to show that the $C^\infty$ homomorphism $\text{pr}_1 : H \to G$ is a diffeomorphism (so its inverse is $C^\infty$). But this latter map is a bijective Lie group homomorphism, so by an application of Sard’s theorem and homogeneity developed in Exercise 5(iii) of HW3 it is necessarily a diffeomorphism! (Beware that in general a $C^\infty$ homeomorphism between smooth manifolds need not be a diffeomorphism, as illustrated by $x \mapsto x^3$ on $\mathbb{R}$.)

2. Fundamental Theorem of Algebra

Now we present a beautiful proof of the Fundamental Theorem of Algebra that was discovered by Witt (and rediscovered by Benedict Gross when he was a graduate student). Let $F$ be a finite extension of $\mathbb{C}$ with degree $d \geq 1$. Viewing $F$ as an $\mathbb{R}$-vector space, $F \simeq \mathbb{R}^{2d}$ with $2d \geq 2$. Our aim is to prove $d = 1$.

Topologically, $F - \{0\} = \mathbb{R}^{2d} - \{0\}$ is connected. For any finite-dimensional associative $\mathbb{R}$-algebra (equipped with its natural manifold structure as an $\mathbb{R}$-vector space), the open submanifold of units is a Lie group. So $F^\times$ is a connected commutative Lie group.

In class we have seen via the exponential map that every connected commutative Lie group is a product of copies of $S^1$ and $\mathbb{R}$. Thus, $F^\times \simeq (S^1)^r \times \mathbb{R}^{2d-r}$ as Lie groups, for some $0 \leq r \leq 2d$. Necessarily $r > 0$, as the vector space factor $\mathbb{R}^{2d-r}$ is torsion-free whereas $F^\times$ has nontrivial torsion (any of the nontrivial roots of unity in $\mathbb{C}^\times$). By contracting the factor $\mathbb{R}^{2d-r}$ to a point, we see that $F^\times$ retracts onto $(S^1)^r$, and so by the homotopy-invariance and direct product functoriality of the fundamental group it follows that $\pi_1(F^\times) = \pi_1((S^1)^r)) = \pi_1(S^1)^r = \mathbb{Z}^r \neq 0$. But we noted that $F^\times = \mathbb{R}^{2d} - \{0\}$, and for $n > 1$ the group $\pi_1(\mathbb{R}^n - \{0\})$ is nontrivial only for $n = 2$, so $2d = 2$ and hence $d = 1$. 

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