Math 210C. The “Weyl Jacobian” formula

1. Introduction

Let \( G \) be a connected compact Lie group, and \( T \) a maximal torus in \( G \). Let \( q : (G/T) \times T \to G \) be the map \((\bar{\vartheta}, t) \mapsto g t g^{-1}\). Fix choices of nonzero left-invariant top-degree differential forms \( d\vartheta \) on \( G \) and \( dt \) on \( T \), and let \( d\bar{\vartheta} \) be the associated left-invariant top-degree differential form on \( G/T \), determined by the property that
\[
d\bar{\vartheta}(e) \otimes dt(\bar{\vartheta}) = dg(e)
\]
via the canonical isomorphism \( \det(\text{Tan}_T^*(G/T)) \otimes \det(\text{Tan}_T^*(T)) \simeq \det(\text{Tan}_T^*(G)) \). These differential forms are used to define orientations on \( G, T \), and \( G/T \), hence also on \((G/T) \times T\) via the product orientation.

Remark 1.1. Recall that for oriented smooth manifolds \( X \) and \( X' \) with respective dimensions \( d \) and \( d' \), the product orientation on \( X \times X' \) declares the positive ordered bases of \( T_{(x,x')}(X \times X') = T_x(X) \oplus T_{x'}(X') \) to be the ones in the common orientation class of the ordered bases \( \{v_1, \ldots, v_d, v'_1, \ldots, v'_{d'}\} \), where \( \{v_i\} \) is an oriented basis of \( T_x(X) \) and \( \{v'_j\} \) is an oriented basis of \( T_{x'}(X') \). If we swap the order of the factors (i.e., we make an ordered basis for \( T_{(x,x')}(X \times X') \) by putting the \( v'_j \)'s ahead of the \( v_i \)'s) then the orientation on \( X \times X' \) changes by a sign of \((-1)^{dd'}\). Consequently, as long as one of \( d \) or \( d' \) is even, there is no confusion about the orientation on \( X \times X' \). Fortunately, it will turn out that \( G/T \) has even dimension!

Since \( dt \wedge d\bar{\vartheta} \) is a nowhere-vanishing top-degree \( C^\infty \) differential form on \((G/T) \times T\), there is a unique \( C^\infty \) function \( F \) on \((G/T) \times T\) satisfying
\[
q^*(dg) = F \cdot dt \wedge d\bar{\vartheta}.
\]
(In class we used \( d\bar{\vartheta} \wedge dt \), as in the course text; this discrepancy will not matter since we’ll eventually show that \( \dim(G/T) \) is even.) Let’s describe the meaning of \( F(\bar{\vartheta}_0, t_0) \in \mathbb{R} \) as a Jacobian determinant. For any point \((\bar{\vartheta}_0, t_0) \in (G/T) \times T\), we may and do choose an oriented ordered basis of \( \text{Tan}_{t_0}(T) \) whose ordered dual basis has wedge product equal to \( dt(t_0) \). We also may and do choose an oriented ordered basis of \( \text{Tan}_{\bar{\vartheta}_0}(G/T) \) whose ordered dual basis has wedge product equal to \( d\bar{\vartheta}(\vartheta_0) \). Use these to define an oriented basis of \( \text{Tan}_q(\bar{\vartheta}_0, t_0)((G/T) \times T) \) by following the convention for product orientation putting \( T \) ahead of \( G/T \). Finally, choose an oriented ordered basis of \( \text{Tan}_q(\bar{\vartheta}_0, t_0)(G) \) whose associated ordered dual basis has wedge product equal to \( dq(q(\vartheta_0, t_0)) \).

Consider the matrix of the linear map
\[
dq(\vartheta_0, t_0) : \text{Tan}_{\bar{\vartheta}_0, t_0}((G/T) \times T) \to \text{Tan}_q(\bar{\vartheta}_0, t_0)(G)
\]
relative to the specified ordered bases on the source and target. The determinant of this matrix is exactly \( F(\vartheta_0, t_0) \). (Check this!) In particular, \( F(\vartheta_0, t_0) \neq 0 \) precisely when \( q \) is a local \( C^\infty \) isomorphism near \((\vartheta_0, t_0)\), and \( F(\vartheta_0, t_0) > 0 \) precisely when \( q \) is an orientation-preserving local \( C^\infty \) isomorphism near \((\vartheta_0, t_0)\). Provided that \( \dim(G/T) \) is even, it won’t matter which way we impose the orientation on \((G/T) \times T\) (i.e., which factor we put “first”).

In view of the preceding discussion, it is reasonable to introduce a suggestive notation for \( F \): we shall write \( \det(dq(\vartheta, t)) \) rather than \( F(\vartheta, t) \). Of course, this depends on more than
just the linear map $dq(\bar{g}, t)$: it also depends on the initial choices of invariant differential forms $dg$ on $G$ and $dt$ on $T$, as well as the convention to orient $(G/T) \times T$ by putting $T$ ahead of $G/T$ (the latter convention becoming irrelevant once we establish that dim$(G/T)$ is even). Such dependence is suppressed in the notation, but should not be forgotten.

The purpose of this handout is to establish an explicit formula for the Jacobian determinant $\det(dq)$ associated to the map $q$. It is expressed in terms of the map $\text{Ad}_{G/T} : T \to \text{GL} (\text{Tan}_e(G/T))$.

so let’s recall the definition of this latter map more generally (as given in dual form in Exercise 2 of HW5).

If $H$ is a closed subgroup of a Lie group $G$ then $\text{Ad}_{G/H} : H \to \text{GL} (\text{Tan}_e(G/H))$ is the $C^\infty$ homomorphism that carries $h \in H$ to the linear automorphism $d(\ell_h)(e)$ of $\text{Tan}_e(G/H) = \mathfrak{g}/\mathfrak{h}$ arising from the left translation on $G/H$ by $h$ (which fixes the coset $\bar{e} = \{H\}$ of $h$). Note that it is equivalent to consider the effect on $\mathfrak{g}/\mathfrak{h}$ of the conjugation map $c_h : x \mapsto hxh^{-1}$ since right-translation on $G$ by $h^{-1}$ has no effect upon passage to $G/H$. Put in other terms, $\text{Ad}_{G/H}(h)$ is the effect on $\mathfrak{g}/\mathfrak{h}$ of the automorphism $d_c(h)(e) = \text{Ad}_G(h)$ of $\mathfrak{g}$ that preserves $\mathfrak{h}$ (since $\text{Ad}_G(h)|_\mathfrak{h} = \text{Ad}_H(h)$ due to functoriality in the Lie group for the adjoint representation).

2. Main result

We shall prove the following formula for the “Weyl Jacobian” $\det(dq)$:

**Theorem 2.1.** For any $(\bar{g}, t) \in (G/T) \times T$,

$$\det(dq(\bar{g}, t)) = \det(\text{Ad}_{G/T}(t^{-1}) - 1).$$

In class we will show that dim$(G/T)$ is even, so there is no possible sign ambiguity in the orientation on $(G/T) \times T$ that underlies the definition of $\det(dq)$. Our computation in the proof of the Theorem will use the orientation which puts $T$ ahead of $G/T$; this is why we used $dt \wedge d\bar{g}$ rather than $d\bar{g} \wedge dt$ when initially defining $\det(dq)$. Only after the evenness of dim$(G/T)$ is proved in class will the Theorem hold without sign ambiguity. (The proof of such evenness will be insensitive to such sign problems.)

**Proof.** We first use left translation to pass to the case $\bar{g} = \bar{e}$, as follows. For $g_0 \in G$, we have a commutative diagram

$$
\begin{array}{ccc}
(G/T) \times T & \xrightarrow{q} & G \\
\downarrow{\lambda} & & \downarrow{c_{g_0}} \\
(G/T) \times T & \xrightarrow{q} & G
\end{array}
$$

where $\lambda(\bar{g}, t) = (g_0, \bar{g}, t)$. The left-translation by $g_0$ on $G/T$ pulls $d\bar{g}$ back to itself due to left invariance of this differential form on $G/T$. The conjugation $c_{g_0}$ pulls $dg$ back to itself because $dg$ is bi-invariant (i.e., also right-invariant) due to the triviality of the algebraic modulus character $\Delta_G$ (since $G$ is compact and connected). It therefore follows via the definition of $\det(dq)$ that

$$\det(dq(g_0, \bar{g}, t)) = \det(dq(\bar{g}, t)).$$
Hence, by choose $g_0$ to represent the inverse of a representative of $\overline{g}$, we may and do restrict attention to the case $\overline{g} = e$.

Our aim is to show that for any $t \in T$, $\det(dq(I, t_0)) = \det(\Ad_G(T(t_0^{-1}) - 1)$ for all $t_0 \in T$. Consider the composite map

$$f : (G/T) \times T \xrightarrow{1 \times t_0} (G/T) \times T \xrightarrow{q} G \xrightarrow{t_0^{-1}} G.$$  

This carries $(\overline{g}, t)$ to $(t_0^{-1}gt_0)(tg^{-1})$ (which visibly depends only on $gT$ rather than on $g$, as it must); this map clearly carries $(\overline{g}, e)$ to $e$. The first and third steps pull the chosen differential forms on $(G/T) \times T$ and $G$ (namely, $dt \wedge d\overline{g}$ and $dg$) back to themselves due to the arranged left-invariance properties. Consequently, $\det(dq(\overline{g}, t_0)) = \det(df(\overline{g}, e))$, where the “determinant” of

$$df(\overline{g}, e) : (g/t) \oplus t \to g$$

is defined using oriented ordered bases of $t$, $g/t$, and $g$ whose wedge products are respectively dual to $dt(e)$, $d\overline{g}(\overline{e})$, and $dg(e)$ (and the direct sum is oriented by putting $t$ ahead of $g/t$).

The definition of $d\overline{g}(\overline{e})$ uses several pieces of data: $dg(e)$, $dt(e)$, and the natural isomorphism $\det(V') \otimes \det(V'') \cong \det(V)$ associated to a short exact sequence of finite-dimensional vector spaces $0 \to V' \to V \to V'' \to 0$ (applied to $0 \to t \to g \to \Tan_t(G/T) \to 0$). Thus, we can choose the oriented ordered basis of $\Tan_t(G/T) = g/t$ adapted to $d\overline{g}(\overline{e})$ to be induced by an oriented ordered basis of $g$ adapted to $dg(e)$ that has as its initial part an ordered oriented basis of $t$ adapted to $dt(e)$. To summarize, the matrix for $df(\overline{g}, e) : (g/t) \oplus t \to g$ rests on: an ordered basis of $t$, an extension of this to an ordered basis of $g$ by appending additional vectors at the end of the ordered list, and the resulting quotient ordered basis of $g/t$.

The restriction $df(\overline{g}, e)|_t$ to the direct summand $t$ is the differential of $f(\overline{g}, \cdot) : T \to G$ that sends $t \in T$ to $(t_0^{-1}et_0)t^{-1} = t \in G$. In other words, this restriction is the natural inclusion of $t$ into $g$.

Composing the restriction $df(\overline{g}, e)|_{g/t}$ to the direct summand $g/t$ with the quotient map $g \to g/t$ is the endomorphism of $g/t$ that is the differential at $\overline{e}$ of the map $k : G/T \to G/T$ defined by

$$(g) \mapsto (t_0^{-1}gt_0)eg^{-1} \mod T = c_{t_0^{-1}}(g)g^{-1} \mod T = m(c_{t_0^{-1}}(g), g^{-1}) \mod T.$$  

Since the group law $m : G \times G \to G$ has differential at $(e, e)$ equal to addition in $g$, and inversion $G \to G$ has differential at the identity equal to negation on $g$, clearly

$$dk(\overline{e}) = (\Ad_G(t_0^{-1}) \mod t) - 1 = \Ad_G(t_0^{-1}) - 1.$$  

Our ordered basis of $g$ begins with an ordered basis of $t$ and the remaining part lifts our chosen ordered basis of $g/t$, so the matrix used for $df(\overline{g}, e)$ has the upper triangular form

$$\begin{pmatrix} 1 & * \\ 0 & M(t_0) \end{pmatrix}$$

where the lower-right square $M(t_0)$ is the matrix of the endomorphism $\Ad_{G/T}(t_0^{-1}) - 1$ of $g/t$ relative to the same ordered basis on its source and target. Consequently, the determinant of this matrix for $df(\overline{g}, e)$ is equal to $\det M(t_0) = \det(\Ad_{G/T}(t_0^{-1}) - 1)$ (the intrinsic determinant using any fixed choice of ordered basis for the common source and target $g/t$).