Let $G$ be a Lie group. One of the most basic tools in the investigation of the structure of $G$ is the conjugation action of $G$ on itself: for $g \in G$ we define $c_g : G \to G$ to be the $C^\infty$ automorphism $x \mapsto gxg^{-1}$. (This is not interesting when $G$ is commutative, but we will see later that connected commutative Lie groups have a rather simple form in general.)

The adjoint representation of $G$ on its tangent space $\mathfrak{g} = T_e(G)$ at the identity is the homomorphism

$$\text{Ad}_G : G \to \text{GL}(\mathfrak{g})$$

defined by $\text{Ad}_G(g) = dc_g(e)$. This is a homomorphism due to the Chain Rule: since $c_{g'} \circ c_g = c_{g'g}$ and $c_g(e) = e$, we have

$$\text{Ad}_G(g'g) = dc_{g'}(e) \circ dc_g(e) = \text{Ad}_G(g') \circ \text{Ad}_G(g).$$

In this handout we prove the smoothness of $\text{Ad}_G$ (which the course text seems to have overlooked), compute the derivative

$$d(\text{Ad}_G)(e) : \mathfrak{g} \to T_e(\text{GL}(\mathfrak{g})) = \text{End}(\mathfrak{g})$$

at the identity, and use $\text{Ad}_G$ to establish a very useful formula relating the Lie bracket rather directly to the group law on $G$ near $e$.

1. Smoothness and examples

To get a feeling for the adjoint representation, let’s consider the case $G = \text{GL}_n(\mathbb{R})$. For any $X \in \mathfrak{g} = \text{Mat}_n(\mathbb{R})$, a parametric curve in $G$ through the identity with velocity vector $X$ at $t = 0$ is $\alpha_X(t) := \exp(tX)$. Thus, the differential $\text{Ad}_G(g) = dc_g(e)$ sends $X = \alpha'_X(0)$ to the velocity at $t = 0$ of the parametric curve

$$c_g \circ \alpha_X : t \mapsto g \exp(tX)g^{-1} = 1 + gtXg^{-1} + \sum_{j \geq 2} \frac{t^j}{j!} gX^j g^{-1},$$

so clearly this has velocity $gXg^{-1}$ at $t = 0$. In other words, $\text{Ad}_G(g)$ is $g$-conjugation on $\text{Mat}_n(\mathbb{R})$. This is visibly smooth in $g$.

We can use a similar parametric curve method to compute $d(\text{Ad}_G)(e)$ for $G = \text{GL}_n(\mathbb{R})$, as follows. Choose $X \in \mathfrak{g}$, so $\alpha_X(t) := \exp(tX)$ is a parametric curve in $G$ with $\alpha'_X(0) = X$. Hence, $d(\text{Ad}_G)(e)(X)$ is the velocity at $t = 0$ of the parametric curve $\text{Ad}_G(\exp(tX)) \in \text{GL}(\mathfrak{g})$. In other words, it is the derivative at $t = 0$ of the parametric curve $c_{\exp(tX)} \in \text{GL}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$.

For $Y \in \mathfrak{g}$,

$$\exp(tX) \circ \exp(-tX) = (1 + tX + t^2(\cdot)) \circ Y(1 - tX + t^2(\cdot)) = (Y + tXY + t^2(\cdot)) \circ (1 - tX + t^2(\cdot)),$$

and this is equal to $Y + t(XY - XY) + t^2(\cdot \cdot \cdot)$, so its $\text{End}(\mathfrak{g})$-valued velocity vector at $t = 0$ is the usual commutator $XY - YX$ that we know to be the Lie bracket on $\mathfrak{g}$.

Next we take up the proof of smoothness in general. First, we localize the problem near the identity using the elementary:

**Lemma 1.1.** Let $G$ and $H$ be Lie groups. A homomorphism of groups $f : G \to H$ is continuous if it is continuous at the identity, and it is $C^\infty$ if it is $C^\infty$ near the identity.
Proof. The left-translation $\ell_g : G \to G$ is a homeomorphism carrying $e$ to $g$, and likewise $\ell_{f(g)} : H \to H$ is a homeomorphism. Since

$$f \circ \ell_g = \ell_{f(g)} \circ f$$

(as $f$ is a homomorphism), continuity of $f$ at $g$ is equivalent to continuity of $f$ at $e$. This settles the continuity aspect. In a similar manner, since left translations are diffeomorphisms and $\ell_g$ carries an open neighborhood of $e$ onto one around $g$ (and similarly for $\ell_{f(g)}$ on $H$), if $f$ is $C^\infty$ on an open $U$ around $e$ then $f$ is also $C^\infty$ on the open $\ell_g(U)$ around $g$ due to (1.1). Since the $C^\infty$-property is local on $G$, it holds for $f$ if it does so on an open set around every point.

Finally, we prove smoothness of $\text{Ad}_G$. Since the conjugation-action map $c : G \times G \to G$ defined by $(g, g') \mapsto gg'g^{-1}$ is $C^\infty$ and $c(e, e) = e$, we can choose a open coordinate domains $U \subset U' \subset G$ around $e$ so that $c(U \times U) \subset U'$. Let $\{x_1, \ldots, x_n\}$ be a coordinate system on $U'$ with $x_i(e) = 0$, and define $f_i = x_i \circ c : U \times U \to \mathbb{R}$ as a function on $U \times U \subset \mathbb{R}^{2n}$. Let $\{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ denote the resulting product coordinate system on $U \times U$.

Each $f_i$ is smooth and $c_g : U \to U'$ has $i$th component function $f_i(g, z_1, \ldots, z_n)$ with $g \in U$. Thus, the matrix $\text{Ad}_G(g) = d(c_g)(e) \in \text{Mat}_n(\mathbb{R})$ has $ij$-entry equal to $(\partial f_i/\partial z_j)(0)$. Hence, smoothness of $\text{Ad}_G$ on $U$ reduces to the evident smoothness of each $\partial f_i/\partial z_j$ in the first $n$ coordinates $y_1, \ldots, y_n$ on $U \times U$ (after specializing the second factor $U$ to $e$). By the preceding Lemma, this smoothness on $U$ propagates to smoothness for $\text{Ad}_G$ on the entirety of $G$ since $\text{Ad}_G$ is a homomorphism.

2. Key formula for the Lie bracket

For our Lie group $G$, choose $X,Y \in \mathfrak{g}$. In class we mentioned the fact (to be proved next time) that there is a unique Lie group homomorphism $\alpha_X : \mathbb{R} \to G$ satisfying $\alpha_X'(0) = X$. The automorphism $\text{Ad}_G(\alpha_X(t))$ of $\mathfrak{g}$ therefore makes sense and as a point in the open subset $\text{GL}(\mathfrak{g})$ of $\text{End}(\mathfrak{g})$ it depends smoothly on $t$ (since $\text{Ad}_G$ is smooth). Evaluation on $Y$ for this matrix-valued path defines a smooth path

$$t \mapsto \text{Ad}_G(\alpha_X(t))(Y)$$

valued in $\mathfrak{g}$. We claim that the velocity of this latter path at $t = 0$ is $[X,Y]$. In other words:

**Theorem 2.1.** For any $X,Y \in \mathfrak{g}$,

$$[X,Y] = \frac{d}{dt}_{t=0}(\text{Ad}_G(\alpha_X(t))(Y)).$$

Observe that the left side uses the construction of global left-invariant differential operators whereas the right side is defined in a much more localized manner near $e$. The “usual” proof of this theorem uses the notion of Lie derivative, but the approach we use avoids that.

Proof. Since $Y$ is the velocity at $s = 0$ of the parametric curve $\alpha_Y(s)$, for any $g \in G$ the vector $\text{Ad}_G(g)(Y) = d(c_g)(e)(Y) \in \mathfrak{g}$ is the velocity at $s = 0$ of the parametric curve $c_g(\alpha_Y(s))$. Thus, for any $t$, $\text{Ad}_G(\alpha_X(t))(Y)$ is the velocity at $s = 0$ of

$$a(t,s) := c_{\alpha_X(t)}(\alpha_Y(s)) = \alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$
Note that $a(t, 0) = e$ for all $t$, so for each $t$ the velocity to $s \mapsto a(t, s) \in G$ lies in $T_e(G) = \mathfrak{g}$; this velocity is nothing other than $\text{Ad}_G(\alpha_X(t))(Y)$, but we shall suggestively denote it as $\frac{d}{ds}|_{s=0}a(t, s)$. Since this is a parametric curve valued in $\mathfrak{g}$, we can recast our problem as proving the identity

$$ [X, Y] = \frac{d}{dt}|_{t=0} \frac{d}{ds}|_{s=0} a(t, s) $$

where $a(t, s) = \alpha_X(t)\alpha_Y(s)\alpha_X(-t)$. We shall compute each side as a point-derivation at $e$ on a smooth function $\varphi$ on $G$ and get the same result.

For the right side, an exercise to appear in HW3 (Exercise 9 in I.2) shows that its value on $\varphi$ is the ordinary 2nd-order multivariable calculus derivative

$$ \frac{\partial^2}{\partial t \partial s}|_{(0,0)} \varphi(a(t, s)) $$

of the smooth function $\varphi \circ a : \mathbb{R}^2 \to \mathbb{R}$. By a clever application of the Chain Rule, it is shown in the course text (on page 19, up to swapping the roles of the letters $s$ and $t$) that this 2nd-order partial derivative is equal to the difference

$$ \frac{\partial^2}{\partial t \partial s}|_{(0,0)} \varphi(\alpha_X(t)\alpha_Y(s)) - \frac{\partial^2}{\partial t \partial s}|_{(0,0)} \varphi(\alpha_Y(s)\alpha_X(t)) $$

Letting $\tilde{X}$ and $\tilde{Y}$ respectively denote the left-invariant vector fields extending $X$ and $Y$ at $e$, we want this difference of 2nd-order partial derivatives to equal the value $[X, Y] = [\tilde{X}, \tilde{Y}](\varphi)(e)$ at $e$ of $\tilde{X}(\tilde{Y}(\varphi)) - \tilde{Y}(\tilde{X}(\varphi))$, so it suffices to prove in general that

$$ \frac{\partial^2}{\partial t \partial s}|_{(0,0)} \varphi(\alpha_X(t)\alpha_Y(s)) = X(\tilde{Y}\varphi) $$

(and then apply this with the roles of $X$ and $Y$ swapped).

In our study next time of the construction of 1-parameter subgroups we will see that for any $g \in G$, $\tilde{Y}(\varphi)(g) = (\partial_s|_{s=0})(\varphi(g\alpha_Y(s)))$. Thus, setting $g = \alpha_X(t)$, for any $t$ the $s$-partial at $s = 0$ of $\varphi(\alpha_X(t)\alpha_Y(s))$ is equal to $\tilde{Y}(\varphi)(\alpha_X(t))$. By the same reasoning now applied to $X$ instead of $Y$, passing to the $t$-derivative at $t = 0$ yields $(\tilde{X}(\tilde{Y}\varphi))(e) = X(\tilde{Y}\varphi))$.

3. Differential of Adjoint

Finally, we connect the Lie bracket to the adjoint representation of $G$:

**Theorem 3.1.** Let $G$ be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then $d(\text{Ad}_G)(e) \in \text{End}(\mathfrak{g})$ is equal to $\text{ad}_g$. In other words, for $X \in \mathfrak{g}$, $d(\text{Ad}_G)(e)(X) = [X, \cdot]$.

**Proof.** Choose $X \in \mathfrak{g}$, so $\alpha_X(t)$ is a parametric curve in $G$ with velocity $X$ at $t = 0$. Consequently, $d(\text{Ad}_G)(e)(X)$ is the velocity vector at $t = 0$ to the parametric curve $\text{Ad}_G(\alpha_X(t))$ valued in the open subset $\text{GL}(\mathfrak{g})$ of $\text{End}(\mathfrak{g})$.

Rather generally, if $L : (-\epsilon, \epsilon) \to \text{End}(V)$ is a parametric curve whose value at $t = 0$ is the identity then for any $v \in V$ the velocity to $t \mapsto L(t)(v)$ at $t = 0$ is $L'(0)(v)$. Indeed, the second-order Taylor expression $L(t) = 1 + tA + t^2B(t)$ for a smooth parametric curve $B(t)$ valued in $\text{End}(V)$ implies that $L(t)(v) = v + tA(v) + t^2B(t)(v)$, so this latter curve valued in $V$ has velocity $A(v)$. But clearly $A = L'(0)$, so our general velocity identity is proved.
Setting $L = \text{Ad}_G \circ \alpha_X$ and $v = Y$, we conclude that $\text{Ad}_G(\alpha_X(t))(Y)$ has velocity at $t = 0$ equal to the evaluation at $Y$ of the velocity at $t = 0$ of the parametric curve $\text{Ad}_G \circ \alpha_X$ valued in $\text{End}(g)$. But by the Chain Rule this latter velocity is equal to 

$$d(\text{Ad}_G)(\alpha_X(0)) \circ \alpha'_X(0) = d(\text{Ad}_G(e))(X),$$

so $d(\text{Ad}_G(e))(X)$ carries $Y$ to the velocity at $t = 0$ that equals $[X, Y]$ in Theorem 2.1.