Localization

Let A be a commutative ring. By a multiplicative set $S \subset A$, we mean a subset such that $1 \in S$ and if $s, t \in S$ then $st \in S$. We will construct a ring $S^{-1}A$ and a homomorphism $i_S : A \to S^{-1}A$ that has the following universal property:

For all $s \in S$, $i_S(s)$ is invertible in $S^{-1}A$, and for every ring homomorphism $\phi : A \to B$ such that $\phi(s)$ is invertible in B for all $s \in S$, there exists a unique homomorphism $\phi_S : S^{-1}A \to B$ such that $\phi = \phi_S \circ i_S : A \to S^{-1}A \to B$.

In the definition of multiplicative set, $0 \in S$ is technically allowed. However, it is very obvious in that case that $S^{-1}A = (0)$, the zero ring, since 0 invertible implies $1 = 0$. This is not very interesting. If $1 \neq 0$ in A, that is, if $A \neq (0)$, and if $0 \notin S$, then $S^{-1}A \neq (0)$ will follow as well.

If A is an integral domain and $0 \notin S$, then $S^{-1}A$ is a rather obvious subring of the field of fractions of A, namely, the subring of fractions whose denominators are elements of S. But even in this case the main point is not just the construction of $S^{-1}A$, but rather the connections between ideals in A and $S^{-1}A$ and the many applications in commutative algebra, number theory, and algebraic geometry.

The usual argument with universal properties shows that if $S^{-1}A$ exists, it is unique up to isomorphism.

The ring $S^{-1}A$ is called a **localization** of A. This terminology arises from consideration of rings of continuous functions, with values, say, in a field. If A is a ring of functions on a space X and if $Y \subset X$ is a subspace, which could be a single point, let $S = S(Y) \subset A$ denote the subset of functions which have no zeros on Y. It is clear that S is a multiplicative set. The quotients a/s, with $a \in A$ and $s \in S$, represent well-defined germs of functions near Y. That is, each such quotient defines a function in some neighborhood of Y, specifically, the neighborhood where s is non-zero. These germs can be added and multiplied and form a ring, whose algebraic properties reflect the properties of X *locally*, that is, near Y. The ring A might have zero divisors, some elements of S might even be zero divisors, but the ring of germs of functions near Y which can be expressed as quotients a/s, still makes sense. It is perhaps useful to keep this picture in mind as motivation for the general construction below.

Before constructing $S^{-1}A$, we list some examples of multiplicative sets.

(i) S is the set of all non-zero divisors in A. If A is an integral domain, this is just the set of all non-zero elements of A, and $S^{-1}A$ will turn out to be the field of fractions of A.

(ii) $S = A - P$, where $P \subset A$ is a prime ideal. In this situation, the localization $S^{-1}A$ is often denoted $A_{(P)}$, and referred to as the localization of A at the prime P. The ring $A_{(P)}$ is a local ring, that is, a ring with a unique maximal ideal. If $A = A(V)$ is the affine coordinate ring of a variety over an algebraically closed field K, and if $P = I(p) \subset A$ is the maximal ideal corresponding to a point $p \in V$, then the localization $A_{(P)}$ is a ring of germs of K-valued functions defined on open neighborhoods of $p \in V$ in the Zariski topology. (It is a little harder to interpret the localization $A_{(P)}$ as a ring of functions if $P = I(W) \subset A$ is a prime ideal corresponding to an irreducible subvariety $W \subset V$.)

(iii) It is obvious that arbitrary intersections of multiplicative sets are multiplicative sets. For example, if $\{P_j\}$ is a family of prime ideals in A, then $\cap_j (A - P_j) = A - \cup_j P_j$ is a multiplicative set.

(iv) If $s \in A$ is an element which is not nilpotent, then $S = \{s^n \mid n \geq 0\}$ is a multiplicative set.

Suppose $\phi : A \to B$ is a ring homomorphism so that some element $s \in S$ is a zero divisor in A and $\phi(s) \in B$ is invertible. If $as = 0 \in A$ then obviously $\phi(a) = 0 \in B$, since $\phi(a)\phi(s) = 0$. This observation makes it clear that $i_S : A \to S^{-1}A$ will not be injective if any elements of S are zero-divisors in A. A consequence is that the construction of $S^{-1}A$ is somewhat more subtle than, say, the construction of the field of fractions of an integral domain. On the other hand, this observation about consequences of the existence of zero divisors in S motivates the precise definition of $S^{-1}A$ in general, which we now give.

$S^{-1}A = \{[a/s] \mid a \in A, s \in S\}/\sim$, where $[a'/s'] \sim [a''/s'']$ if and only if there exists $s \in S$ with $a's''s = a''s's \in A$.

The homomorphism $i_S : A \to S^{-1}A$ will be defined by $i_S(a) = [a/1]$. The symbol $[a/s]$ will name the element $i_S(a)i_S(s)^{-1} \in S^{-1}A$. So we certainly want $[a'/s'] = [a's''/s's''] = [a's''s/s's''s]$, and $[a''/s''] = [a''s'/s''s'] = [a''s's/s''s's]$. The usual definition of equality of fractions, $a'/s' = a''/s''$ if $a's'' = a''s'$, would be adequate if no element of S were a zero divisor. The additional factor of s in the definition above allows for the possibility that $a's'' - a''s' \neq 0$, but $a's''s - a''s's = 0$ for some $s \in S$.

Exercise 1 *Define addition and multiplication in $S^{-1}A$, and verify that these operations are well-defined. Acknowledge that you ought to also check all ring axioms, associative laws, distributive laws, identity elements, etc, but don't bother with all that.*

Exercise 2 *Verify the universal property, stated at the outset, for $i_S : A \to S^{-1}A$.*

Exercise 3 *Characterize the set of all elements $t \in A$ such that $i_S(t)$ is invertible in $S^{-1}A$. (This subset of A is also a multiplicative set, called the **saturation of S in A**. Localizing A with respect to the saturation of S yields the same ring as localizing with respect to S. This can be seen instantly, since the universal property of $S^{-1}A$ implies that this ring also has the required universal property to be the localization with respect to the saturation of S.)*

Next, we investigate relations between localization and ideals. Suppose $I \subset A$ is an ideal with $I \cap S = \emptyset$. Let $A^* = A/I$ and let S^* be the image of S in A^*. Then S^* is a multiplicative set in A^*. The following exercise expresses the commutativity of localization and residue ring formation. One can either divide by an ideal first and then localize, or localize first and then divide by an appropriate ideal.

Exercise 4 $(S^*)^{-1}A^* \cong S^{-1}A/IS^{-1}A$. *(Although it is easy enough to establish one-to-one ring homomorphisms in both directions, it is more elegant to verify that the ring on the right has the universal property of the ring on the left, hence it is isomorphic to the ring on the left.)*

If $\phi : A \to B$ is any ring homomorphism there are operations known as contraction and extension which relate ideals in B and A. If $J \subset B$ is an ideal, the contraction is $J^c = \phi^{-1}(J) \subset A$, the inverse image of J in A. If $I \subset A$ is an ideal, the extension is $I^e = \phi(I)B$, the ideal generated by the image of I in B. Both the contraction and extension operations preserve inclusions, that is, $J_1 \subset J_2$ implies $J_1^c \subset J_2^c$, and similarly for extension. There are also obvious inclusions $J^{ce} \subset J$ and $I \subset I^{ec}$, from which follow trivially the relations $J^{cec} = J^c$ and $I^{ece} = I^e$. The contraction operation has a number of reasonable properties, but very little can be said in general about extension. For example, the following properties of contraction are all easy to check, while the analogues for extension are all false.

(i) If J is prime, primary, or radical, then J^c is, respectively, prime, primary, or radical.

(ii) $\operatorname{rad} J^c = (\operatorname{rad} J)^c$ and $\cap(J_i^c) = (\cap J_i)^c$.

In case the homomorphism is a localization $i_S : A \to S^{-1}A$, one can say more about the contraction operation, and, in addition, the extension operation has many good properties.

Exercise 5 *If $J_S \subset S^{-1}A$ is an ideal, then $J_S^{ce} = J_S$. If $I \subset A$ is an ideal, then $I^e = \{[a/s] \mid a \in I, s \in S\} \subset S^{-1}A$ and $I^{ec} = \{b \in A \mid sb \in I, \text{for some } s \in S\}$.*

Exercise 6 *$I^e \subset S^{-1}A$ is proper if and only if $I \cap S = \emptyset$. If $I \subset A$ is prime or primary and $I \cap S = \emptyset$, then $I^e \subset S^{-1}A$ is, respectively, prime or primary. Moreover, in each of these cases, $I^{ec} = I \subset A$.*

Exercise 7 *If $I \subset A$ is any ideal, then* $\operatorname{rad} I^e = (\operatorname{rad} I)^e \subset S^{-1}A$. *If I_1 and I_2 are ideals in A, then $I_1^e \cap I_2^e = (I_1 \cap I_2)^e \subset S^{-1}A$.*

It follows from Exercises 5 and 6 that contraction and extension define inclusion preserving bijections between the sets of all prime or primary ideals, respectively, in $S^{-1}A$ and the sets of prime or primary ideals in A which are disjoint from S. It also follows, from Exercises 6 and 7, that a primary decomposition of an ideal $I \subset A$ determines, by extension, a primary decomposition of $I^e \subset S^{-1}A$. In this extension, any of the primary components of I that are not disjoint from S will generate the unit ideal in $S^{-1}A$, hence disappear from the primary decomposition of I^e.

Another simple consequence of the correspondence between prime ideals of $S^{-1}A$ and prime ideals of A disjoint from S is that in the case $S = A - P$, where P is a prime ideal of A, the localization $S^{-1}A = A_{(P)}$ is a local ring. Namely, the unique maximal ideal is $P^e = \{[x/s] \mid x \in P, s \in S\} \subset S^{-1}A$, since, obviously, $P \subset A$ contains all prime ideals of A disjoint from $A - P$. It is also easy enough to verify directly that $A_{(P)}$ is a local ring, since any element not in P^e is clearly a unit in $A_{(P)}$, as it can be written $[t/s]$, with both t and s not in P.

The localization construction is also very important for modules. If $S \subset A$ is a multiplicative set and if M is an A-module, we construct an $S^{-1}A$-module, $S^{-1}M$, and an A-module homomorphism $f_S : M \to S^{-1}M$, with the following universal property:

For every $S^{-1}A$-module N_S and every A-module homomorphism $\phi : M \to N_S$, there exists a unique $S^{-1}A$-module homomorphism $\phi_S : S^{-1}M \to N_S$ such that $\phi = \phi_S \circ f_S : M \to S^{-1}M \to N_S$.

(We point out that $S^{-1}A$ modules are also A-modules, via the ring homomorphism $i_S : A \to S^{-1}A$.)

The definition of $S^{-1}M$, as well as the sum operation and scalar multiplication by elements of $S^{-1}A$, and the proofs that these operations are well-defined, proceeds identically, symbol by symbol, with the construction of $S^{-1}A$ and the ring operations in $S^{-1}A$.

$S^{-1}M = \{[m/s] \mid m \in M, s \in S\}/\sim$, where $[m'/s'] = [m''/s'']$ if and only if there exists $s \in S$ with $m's''s = m''s's$.

The map $f_S : M \to S^{-1}M$ is given by $f_S(m) = [m/1]$. The symbol $[m/s]$ is interpreted as $[1/s]f_S(m)$, in the $S^{-1}A$-module structure on $S^{-1}M$. If you want to strengthen those brain wiring connections, go through the definition of the sum and scalar operations in $S^{-1}M$, the proof that they are well-defined, and the proof that $S^{-1}M$ has the universal property stated above. Of course, the universal property characterizes $S^{-1}M$ uniquely, up to isomorphism as an $S^{-1}A$-module.

Exercise 8 *Prove that localization is an exact functor of modules. That is, an A-module homomorphism $M \to M'$ induces an $S^{-1}A$-module homomorphism $S^{-1}M \to S^{-1}M'$, with the usual functorial properties, and, if $M \to M' \to M''$ is an exact sequence of A-modules, then $S^{-1}M \to S^{-1}M' \to S^{-1}M''$ is an exact sequence of $S^{-1}A$-modules.*