# ARTINIAN RINGS AND MODULES

Let $R$ be a ring (not necessarily commutative) and let $M$ be a left (or right) $R$-module. Then $M$ *Artinian* means that every simple descending chain of submodules $M_1 \supset M_2 \supset M_3 \supset \cdots$ stabilizes, that is, for some $r$ and all $n \geq 0$, $M_r = M_{r+n}$. Equivalently, every non-empty family of submodules of $M$ contains members that are minimal in that family. We say that $R$ is left (or right) Artinian if it is Artinian as a left (or right) module over itself.

To some extent, arguments with Artinian modules are very similar to arguments with Noetherian modules, that is, modules in which every simple ascending chain stabilizes. For example:

LEMMA 1. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $R$-modules, then $M$ is Artinian [resp. Noetherian] if and only if both $M', M''$ are Artinian [resp. Noetherian].

PROOF. For the 'if' direction in both cases, consider a chain of submodules $M_m$ of $M$. Projecting to $M''$, the image chain stabilizes, say from the image of $M_r$. From this point, the kernels of $M_m \to M''$, $m \geq r$, form a chain of submodules of $M'$ that stabilizes. The 5-lemma then implies the $M_m$ stabilize as well.

If $M$ satisfies a chain condition then obviously so does $M'$. But also, given a chain in $M''$, the inverse images in $M$ form a chain. Since $M \to M''$ is surjective, the fact that the chain in $M$ stabilizes implies the original chain in $M''$ stabilizes.

LEMMA 2. Consider a finite chain of submodules of $M$, say $(0) = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_s = M$. Then $M$ is Artinian [resp. Noetherian] if and only if each quotient module $M_{j+1}/M_j$ is Artinian [resp. Noetherian].

PROOF. Apply Lemma 1 inductively for $j \geq 0$ to the sequences

$$0 \to M_j \to M_{j+1} \to M_{j+1}/M_j \to 0.$$

EXAMPLES(i). Modules over any ring that are also finite dimensional vector spaces over a field, and for which all submodules are vector subspaces, are clearly both Artinian and Noetherian. In fact, for vector spaces, finite dimensionality is equivalent to either chain condition separately.

(ii). Non-commutative examples of type (i) include matrix rings over a field or a division algebra and group rings of finite groups over a field. Commutative examples include commutative rings $B$ that are finitely generated over a field $k$ and in which every element is algebraic over $k$. Zero divisors and nilpotent elements are allowed here. The conditions are just equivalent to saying $k \subset B$ is a finitely generated integral extension, so $B$ is finitely generated as $k$-module.

(iii). Simple modules $N$, that is modules with no submodules other than (0) and $N$, are both Artinian and Noetherian. By Lemma 2, any module $M$ that admits a finite composition series with simple quotients $M_{j+1}/M_j$ is both Artinian and Noetherian.

(iv). In fact, the converse is true. Starting with such an $M$, choose a minimal (and therefore simple) proper submodule $(0) \subset M_1$. Then choose $M_2$ minimal among submodules containing $M_1$ properly. So $M_2/M_1$ is simple. Continuing, the process must terminate with $M$ after finitely many steps because $M$ is Noetherian. [One can also start with the Noethrian hypohesis and choose a maximal proper submodule of $M$ and work down. The Artinian hypothesis guarantees the process stops at (0) after finitely many steps.]

(v). Finite direct products of Artinian modules or rings are Artinian.

(vi). Suppose $A$ is a commutative local Noetherian ring, with nilpotent maximal ideal $m$. Then $A$ is Artinian. Namely, apply Lemma 2 to the filtration $A \supset m \supset m^2 \supset \cdots \supset m^k = 0$. Each quotient $m^j/m^{j+1}$ is a finite dimensional vector space over the field $A/m$, since all the $m^j$ are finitely generated ideals in $A$. But also, the $A$-submodules of each such quotient are the same thing as $A/m$ sub-vector spaces, hence these quotients are Artinian $A$-modules.

If $A$ is any Noetherian commutative ring and $m \subset A$ is a maximal ideal, then $A/m^k$ is Artinian. This just more or less repeats the paragraph above, after noting that $A/m^k$ is indeed local. Any element of $A$ not in $m$ is invertible modulo $m^k$.

There is a good structure theory for both non-commutative and commutative Artinian rings. Here I will just deal with the commutative case. The theorem is that any commutative Artinian ring is a finite direct product of rings of the type in Example (vi).

LEMMA 3. In a commutative Artinian ring every prime ideal is maximal. Also, there are only finitely many prime ideals.

PROOF. Consider a prime $P \subset A$. Consider $x \notin P$. The power ideals $(x^m)$ decrease, so we get $(x^n) = (x^{n+1})$ for some $n$. Then $x^n = ax^{n+1}$, so $x^n(1-ax) = 0$. But $x^n \notin P$, hence $1-ax \in P$, which implies $A = P+Ax$. Thus $P$ is maximal.

For the second statement, consider an ideal $J$ minimal among all finite products of distinct primes, $J = \prod P_j$. Given any prime $P$, we have $P \supset JP = J = \prod P_j$, the middle relation because of the minimality of $J$. Thus $P$ contains some $P_j$, and since all primes are maximal $P = P_j$.

LEMMA 4. In a commutative Artinian ring, the nilradical $N = \prod P_j = \bigcap P_j$ is a nilpotent ideal. That is, $N^k = (0)$, some $k \geq 1$.

PROOF. Here, the $P_j$ denote the finitely many distinct primes of $A$. The powers of $N$ decrease, so we can find a smallest $k$ so that $NN^k = N^k$. Claim that $N^k = (0)$. If not, consider a minimal non-zero ideal $J$ so that $JN^k \neq (0)$. ($N$ is one such ideal, so the family is non-empty.) Clearly $J = (a)$ must be principal. Then $aNN^k = aN^k \neq (0)$, so by minimality $(a) = aN$. But then

$a = ax$ for some $x \in N$, hence $(1-x)a = 0$. But $(1-x)$ is a unit, since it is contained in no maximal ideals, so we have a contradiction.

LEMMA 5. A commutative Artinian ring is Noetherian.

PROOF. Since $N^k = (0)$, we have (not necessarily distinct) maximal ideals $m_1, m_2, \cdots, m_r$ with $(0) = m_1 m_2 \cdots m_r$. Consider the filtering

$$A \supset m_1 \supset m_1 m_2 \supset \cdots \supset m_1 m_2 \cdots m_r = (0).$$

Since $A$ is Artinian, the $j^{th}$ quotient in this series is Artinian, but is also a vector space over the field $A/m_j$. Thus these vector spaces are finite dimensional, hence these quotients are Noetherian $A$-modules, from which we conclude from Lemma 2 that $A$ is Noetherian.

THEOREM: If $A$ is a commutative Artinian ring with $N^k = (0)$, where $N = \prod P_j$ is the nilradical, then by the Chinese Remainder Theorem

$$A \simeq \prod A/P_j^k,$$

which is seen to be a finite product of local Noetherian rings, each with a nilpotent maximal ideal.

In this formula, one can obviously replace the common exponent $k$ for the various maximal ideals by exponents $k_j$, the smallest integers so that $P_j^{k_j} = P_j^{k_j+1}$.