Math 210B. Mackey theory and applications

1. Mackey's formula

Let G be a finite group, K and H two subgroups, and (W, ρ) a representation of H over a field k. In class we stated the following result and sketched some ideas in the proof; here we give the complete argument.

Proposition 1.1 (Mackey)**.** *The restriction* $\mathrm{Res}_K(\mathrm{Ind}_H^G(\rho))$ *of the* G*-representation* $\mathrm{Ind}_H^G(\rho)$ *to a* K*-representation decomposes as a direct sum*

$$\bigoplus_{\overline{s} \in H \backslash G / K} \mathrm{Ind}_{s^{-1}Hs \cap K}^K(\rho^s)$$

where s *is a representative of the double coset* $\overline{s}$ *and* $\rho^s(x) := \rho(sxs^{-1})$ *for* x *belonging to the subgroup* $s^{-1}Hs \cap K$ *that depends only on* $\overline{s}$ *and not on its chosen representative* s.

Proof. Let $r \in G$ be a representative of a coset class $\overline{r} \in H \backslash G$, so by design

$$\mathrm{Ind}_H^G(\rho) = \bigoplus_{\overline{r} \in H \backslash G} r^{-1}W$$

(where r^{-1} represents $\overline{r}^{-1} \in G/H$, and $r^{-1}W$ is the image under the action of $r^{-1} \in G$ on the canonical subspace $W = [1] \otimes W \subset \mathrm{Ind}_H^G(\rho) = k[G] \otimes_{k[H]} W$). For $k \in K$ we have $kr^{-1}W = (rk^{-1})^{-1}W$, and as k varies such elements rk^{-1} sweep out an entire double coset $\overline{s} \in H \backslash G / K$, or in other words rk^{-1} sweeps out a set of representatives for $H \backslash (HsK)$ for a representative s of $\overline{s}$.

If we collect the summands $r^{-1}W$ for $\overline{r}$ belonging to a common double coset in $H \backslash G / K$, we arrive at a decomposition

$$\mathrm{Ind}_H^G(\rho) = \bigoplus_{\overline{s} \in H \backslash G / K} \left(\bigoplus_{\overline{r} \in H \backslash HsK} r^{-1}W \right)$$

for which the inner direct sum is K-stable. Thus, it suffices to show for a choice of s that the associated inner direct sum is naturally isomorphic as a K-representation to $\mathrm{Ind}_{s^{-1}Hs \cap K}^K(\rho^s)$.

Fix $\overline{s}$ and a representative $s \in \overline{s}$. The K-action on $H \backslash HsK$ through right multiplication is *transitive*, so the K-action on the inner direct sum for $\overline{s}$ transitively permutes the summands with the stabilizer of $s^{-1}W$ equal to the group of elements $k \in K$ such that $ks^{-1}W = s^{-1}W$, which is to say $sks^{-1} \in H$ or equivalently $k \in s^{-1}Hs \cap K$. For any such k, the effect of k on $s^{-1}W$ is given by the recipe

$$s^{-1}w \mapsto ks^{-1}w = s^{-1}(sks^{-1}w) = s^{-1}(\rho^s(k)(w)).$$

In other words, this "$\overline{s}$-part" is precisely the dynamic description of $\mathrm{Ind}_{s^{-1}Hs \cap K}^K(\rho^s)$. ∎

2. Applications

In class we uses Mackey's formula in the case $K = H$ to prove Mackey's irreducibility criterion: for irreducible (W, ρ), $\mathrm{Ind}_H^G(\rho)$ is irreducible if and only if for all $s \in G - H$ the representations ρ and ρ^s of $H_s := s^{-1}Hs \cap H$ have no irreducible consituents in common. We wish to discuss several applications of this criterion, referring to specific textbook references for details of proofs. The purpose of this summary is just to give some illustrations of the utility of the criterion. In all that follows, k is an algebraically closed field of characteristic 0 (or even just $\mathbf{C}$ if you prefer).

Example 2.1. Suppose $G = H \ltimes A$ for an abelian normal subgroup A. It is shown in §8.2 of Serre's book *Linear representations of finite groups* via Mackey's criterion and Frobenius reciprocity that the irreducible representations of G are given uniquely up to isomorphism by the following construction.

Consider a 1-dimensional character $\chi : A \to k^\times$, so

$$A_\chi := \{g \in G \,|\, \chi(gag^{-1}) = \chi(a) \text{ for all } a \in A\}$$

is a subgroup of $G = H \ltimes A$ containing A; as such it must have the form $H' \ltimes A$ for a subgroup $H' \subset H$, and χ naturally extends to a homomorphism $\chi' : H' \ltimes A \to k^\times$ defined by $h'a \mapsto \chi(a)$ (which can be checked to indeed be a homomorphism due to how H' is defined). For any irreducible representation ρ' of H' made into a representation $\widetilde{\rho}'$ of $H' \ltimes A$ via the composition of ρ' with $H' \ltimes A \twoheadrightarrow H'$, the induced representation $\mathrm{Ind}_{H' \ltimes A}^G(\chi' \otimes \widetilde{\rho}')$ turns out to be irreducible, and uniquely determines the data H', χ', ρ' that enter into its construction.

Example 2.2. What are the irreducible representations of $G = \mathrm{GL}_2(\mathbf{F}_q)$? For details on the following, see the entirely self-contained Chapter 2 of the book *Local Langlands Correspondence for* GL(2) by Bushnell and Henniart (don't be scared by the title: Chapter 2 doesn't require knowing about anything connected to the work and ideas of Langlands); this is vastly more illuminating than the presentation in §2 of Chapter XVIII of Lang's book on the same topic.

Consider the upper-triangular subgroup $B \subset G$ ("B" stands for "Borel"), so $B = T \ltimes U$ where T is the diagonal subgroup $\mathbf{F}_q^\times \times \mathbf{F}_q^\times$ and U is the subgroup of unipotent upper-triangular matrices. For any $\chi_1, \chi_2 \in (\mathbf{F}_q^\times)^\wedge$, by composing the quotient map $B \twoheadrightarrow B/U = T = \mathbf{F}_q^\times \times \mathbf{F}_q^\times$ with the character $\chi_1 \otimes \chi_2 : T \to k^\times$ defined by $(t_1, t_2) \mapsto \chi_1(t_1)\chi_2(t_2)$ we get a 1-dimensional character $\chi : B \to k^\times$. These are in fact *all* of the 1-dimensional characters of B when $q \neq 2$ because any such character must kill U if $q \neq 2$. Indeed, the commutativity of $B/U = T$ implies that U contains the commutator subgroup of B, and we claim that it coincides with the commutator subgroup (and thus is killed by any 1-dimensional character of B) when $q \neq 2$. The formula

$$\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & (t-1)x \\ 0 & 1 \end{smallmatrix}\right),$$

using a fixed $t \neq 0, 1$ (possible since $q \neq 2$) and varying $x \neq 0$ shows that every nontrivial element of U is a commutator. Note in contrast that if $q = 2$ then $B = U$ is abelian, since $T = 1$ in such cases (as $\mathbf{F}_2^\times = 1$).

Consider the induction $\mathrm{Ind}_B^G(\chi)$. This has dimension $\#(G/B)$, and $\#(G/B) = q + 1$. (This size can be determined by brute-force computation of $\#G$ and $\#B$, but here is a more illuminating geometric argument: G acts transitively on $\mathbf{P}^1(\mathbf{F}_q)$ by linear fractional transformations akin to the case of $\mathrm{GL}_2(\mathbf{C})$ acting on $\mathbf{P}^1(\mathbf{C})$, with stabilizer B at $\infty = [1, 0] \in \mathbf{P}^1(\mathbf{F}_q)$, so $\#(G/B) = \#\mathbf{P}^1(\mathbf{F}_q) = q + 1$). We will record below the precise sense in which such inductions are usually irreducible, but this sometimes fails. For example, if we take $\chi_1 = \chi_2$ above then $\chi = \chi_1 \circ \det$ on B, so $\widetilde{\chi} := \chi_1 \circ \det$ on G is an extension of χ to the entirety of G, yielding

$$\mathrm{Ind}_B^G(\chi) = \widetilde{\chi} \otimes \mathrm{Ind}_B^G(\mathbf{1}_B),$$

which is reducible since $\mathrm{Ind}_B^G(\mathbf{1}_B)$ of dimension $q + 1$ contains the trivial representation (with multiplicity 1 by Frobenius reciprocity).

In fact, if we view $\mathrm{Ind}_B^G(\mathbf{1}_G)$ as the space of functions $f : G/B = \mathbf{P}^1(\mathbf{F}_q) \to k$ then the line of constant functions is a copy of the trivial representations and (since $\mathrm{char}(k) = 0$) a G-stable complement is given by the space of such functions f with average value equal to 0. The latter subrepresentation is often denoted St_G, and is called the *Steinberg representation* because it is a special case of a general construction of Steinberg (going far beyond the case of GL_2). It turns out (but is not at all obvious) that the q-dimensional St_G is always *irreducible* as a G-representation, even when $q = 2$. Extensive use of Frobenius reciprocity and Mackey's irreducibility criterion yields:

Theorem 2.3. *Let $\chi = \chi_1 \otimes \chi_2 : B \twoheadrightarrow B/U = T = \mathbf{F}_q^\times \times \mathbf{F}_q^\times \to k^\times$ be as above, and define the "Weyl element" $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the normalizer $N_G(T)$ of T in G, so $w^2 = -1$ is central and the composition $(\chi|_T)^w$ of $\chi|_T$ with w-conjugation corresponds to swapping the order of χ_1 and χ_2.*

(i) *The representation $\mathrm{Ind}_B^G(\chi)$ is irreducible if and only if $(\chi|_T)^w \neq \chi|_T$ (i.e., $\chi_1 \neq \chi_2$), and otherwise it is a direct sum $(\chi_1 \circ \det) \oplus (\chi_1 \otimes \mathrm{St}_G)$. As we vary χ, there are no repetitions up to isomorphism in the collection of irreducible constituents of the representations $\mathrm{Ind}_B^G(\chi)$.*

(ii) *The irreducible representations V of G obtained in* (i) *are precisely those that satisfy $V^U = 0$.*

The irreducible representations in (ii), at least away from the 1-dimensional cases, are called *principal series* representation of G. These are the "easy" ones to construct, insofar as they are found using the induction of 1-dimensional representations from the concrete subgroup B (whose importance is best understood in terms of the general structure theory of smooth connected affine group varieties over general fields). What are the others?

That is, how does one build irreducible representations V of G such that $V^U \neq 0$? (We know such V must exist, since U certainly does not act trivially on the faithful representation space $k[G]$ for G.) Such V are called *cuspidal* (for reasons stemming from the theory of modular forms), and their construction is rather more indirect; it rests on working with the subgroup $\mathbf{F}_{q^2}^\times \hookrightarrow G$ (embedding via choosing an $\mathbf{F}_q$-basis of $\mathbf{F}_{q^2}$ and describing $\mathbf{F}_{q^2}^\times$-scaling on $\mathbf{F}_{q^2}$ in terms of invertible 2×2-matrices relative to this basis) instead of with the subgroup $T \subset G$. The subgroup T is often called a "split torus" in G (for reasons related to the theory of compact Lie groups), whereas $\mathbf{F}_{q^2}^\times \subset G$ is called a "non-split torus" (for reasons that lie beyond the scope of this course).