Math 210B. The bar resolution

1. Motivation

Let G be a group. In class we saw that the functorial identification of M^G with $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M)$ for G-modules M (where $\mathbf{Z}$ is viewed as a G-module with trivial G-action) yields a *unique δ-functorial* identification of the δ-functor $\mathrm{H}^\bullet(G, \cdot)$ with $\mathrm{Ext}^\bullet_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$, where this Ext is computed on the category of left $\mathbf{Z}[G]$-modules. (The formalism of Ext as a derived functor of Hom works verbatim on the abelian category of left modules over any associative ring, including the fact that this category admits enough injectives. The existence of enough projectives is easier, via free modules.) In particular, it follows that if we can write down an explicit $\mathbf{Z}[G]$-linear resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

of $\mathbf{Z}$ as a left $\mathbf{Z}[G]$-module with the F_j's all free over $\mathbf{Z}[G]$ then we have as δ-functors

$$\mathrm{H}^n(G, M) \simeq \mathrm{H}^n(\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M))$$

in the sense that for any short exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

the resulting short exact sequence of *complexes* (of abelian groups)

$$0 \to \mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M') \to \mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M) \to \mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M'') \to 0$$

(short exact since each F_j is $\mathbf{Z}[G]$-projective!) yields a long exact homology sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow \mathrm{H}^n(\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M')) & \longrightarrow & \mathrm{H}^n(\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M)) & \longrightarrow & \mathrm{H}^n(\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M'')) \\ & & & & \downarrow \delta \\ & & \cdots & \longleftarrow & \mathrm{H}^{n+1}(\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M')) \end{array}$$

that *is* the long exact sequence in group cohomology.

The aim of this handout is twofold. First, we will construct an explicit such resolution of $\mathbf{Z}$, called the *bar resolution* (for reasons I do not know), and secondly we will work out in terms of this resolution what the connecting maps are in the above exact sequence. In particular, we will make it explicit in low degree (degrees ≤ 2), as this is very useful in applications.

2. The resolution

Define $F_{-1} = \mathbf{Z}$ (with trivial G-action), $F_0 = \mathbf{Z}[G]$ (with standard basis vector sometimes denoted $(\emptyset)$), and for $j > 0$

$$F_j = \bigoplus_{\underline{g} \in G^j} \mathbf{Z}[G](\underline{g}).$$

In other words, F_j is the free left $\mathbf{Z}[G]$-module on the set G^j (with G^0 understood to be the 1-point set $\{\emptyset\}$). We write elements of $\mathbf{Z}[G]$ in the form of finite sums $\sum_{g \in G} c_g[g]$ with $c_g \in \mathbf{Z}$ (vanishing for all but finitely many $g \in G$). In terms of this notation, to define a $\mathbf{Z}[G]$-linear map $\mathrm{d}_j : F_j \to F_{j-1}$ for $j \geq 0$ it is the same to define a map of sets $\Delta_j : G^j \to F_{j-1}$ for each $j \geq 0$ (and then $\mathrm{d}_j(\sum_i x_i(\underline{g}_i)) = \sum_i x_i \Delta_j(\underline{g}_i)$ for finitely many elements $x_i \in \mathbf{Z}[G]$ and $\underline{g}_i \in G^j$).

We define $\Delta_0 : G^0 \to F_{-1} = \mathbf{Z}$ to carry the single element of G^0 to 1, or in other words (recalling that $\mathbf{Z}$ is equipped with the trivial G-action!) the $\mathbf{Z}[G]$-linear map $\mathrm{d}_0 : \mathbf{Z}[G] \to \mathbf{Z}$ is the "augmentation" map

$$\sum_{g\in G} c_g[g] \mapsto \sum_{g\in G} c_g$$

carrying each $[g]$ to 1. We define $\Delta_1 : G^1 \to F_0 = \mathbf{Z}[G]$ to be the map $(g) \mapsto [g] - [1]$ for all $g \in G$, or equivalently

$$\mathrm{d}_1(\sum x_i[g_i]) = \sum x_i([g_i] - [1])$$

for finitely many elements $g_i \in G$ and $x_i \in \mathbf{Z}[G]$. For $j > 1$, we define $\Delta_j : G^j \to F_{j-1}$ to be

$$\Delta_j(g_1, \dots, g_j) = [g_1] \cdot (g_2, \dots, g_j) + \sum_{i=i}^{j-1} (-1)^i (g_1, \dots, g_i g_{i+1}, \dots, g_j) + (-1)^j (g_1, \dots, g_{j-1}) \tag{2.1}$$

for each ordered j-tuple $(g_1, \dots, g_j) \in G^j$.

It is clear that the augmentation map $\mathrm{d}_0 : \mathbf{Z}[G] \to \mathbf{Z}$ is surjective, so there are two things to be checked:

(i) $d_{j-1} \circ d_j = 0$ for all $j > 0$ (i.e., "$\mathrm{d}^2 = 0$"), so $F_\bullet$ is a complex;

(ii) this complex is exact in all degrees (so it is a free $\mathbf{Z}[G]$-module resolution of $F_{-1} = \mathbf{Z}$).

The proof of (i) will be an non-obvious computation with lots of internal cancellations. The proof of (ii) will involve the usual trick of building "homotopy operators" $h_j : F_j \to F_{j+1}$ for all $j \geq -1$ such that $\mathrm{d}_{j+1} \circ h_j + h_{j-1} \circ \mathrm{d}_j = \mathrm{id}_{F_j}$ for all $j \geq 0$ ("$\mathrm{d}h + h\mathrm{d} = 1$"), so exactness will hold in all degrees ≥ 0 (and exactness in degree -1 was already noted, namely the surjectivity of the augmentation map $\mathbf{Z}[G] \to \mathbf{Z}$ for the group ring $\mathbf{Z}[G]$).

Proposition 2.1. *For all* $j \geq 1$, $\mathrm{d}_{j-1} \circ \mathrm{d}_j = 0$.

Proof. We first verify the cases $j \leq 2$ by hand (since the assertions involve F_0, whereas the "uniform" definition of F_i was only for $i > 0$, depending on one's thoughts about the convention for defining G^0 or games with the empty set). Then we handle all $j > 2$ by a uniform argument.

For $j = 1$, we seek to compose $\mathrm{d}_1 : F_1 \to F_0 = \mathbf{Z}[G]$ with the augmentation map $\mathrm{d}_0 : \mathbf{Z}[G] \to \mathbf{Z}$. To prove the vanishing of this $\mathbf{Z}[G]$-linear composite map, it suffices to check on basis elements for F_1, such as the elements (g) for $g \in G$. By definition, for each $g \in G$ we have $\mathrm{d}_1((g)) = [g] - [1] \in \mathbf{Z}[G]$, and this is visibly killed by the augmentation map.

Now consider $j = 2$. Once again it suffice to verify the vanishing of $\mathrm{d}_1 \circ \mathrm{d}_2 : F_2 \to F_0 = \mathbf{Z}[G]$ when restricted to a $\mathbf{Z}[G]$-basis of F_2, such as the elements (g, g') for $g, g' \in G$. By definition $\mathrm{d}_2((g, g')) = \Delta_1(g, g') = [g] \cdot (g') - (gg') + (g)$, and the $\mathbf{Z}[G]$-linear d_1 carries this to

$$\begin{aligned} [g] \cdot \Delta_1(g') - \Delta_1(gg') + \Delta_1(g) &= [g] \cdot ([g'] - [1]) - ([gg'] - [1]) + ([g] - [1]) \\ &= ([gg'] - [g]) - ([gg'] - [1]) + ([g] - [1]) \\ &= 0, \end{aligned}$$

as desired.

Finally, consider $j > 2$. A direct computation is somewhat messy (due to the need to treat various "boundary terms" in a special way), so we shall instead make an isomorphism with another $\mathbf{Z}[G]$-linear complex for which the computation is simpler. Loosely speaking, we switch to the viewpoint of "homogeneous cochains". More precisely, we define the complex $E_\bullet$ with $E_j = \mathbf{Z}[G^{j+1}]$ for $j \leq 0$

(the free abelian group on the set G^{j+1}), equipped with a $\mathbf{Z}[G]$-module structure via the diagonal action:

$$[g]\cdot(g_0,\dots,g_j) = (gg_0,\dots,gg_j).$$

To define an isomorphism of $\mathbf{Z}[G]$-modules $F_j \simeq E_j$, we first note that every element of E_j has the unique form

$$\sum_{g\in G}\sum_{\underline{g}\in G^j} c_{g,\underline{g}}(g,\underline{g}) = \sum_{\underline{g}\in G^j}(\sum_{g\in G} c_{g,\underline{g}}[g])(1,g^{-1}\underline{g})$$

where $c_{g,\underline{g}} = 0$ for all but finitely many pairs $(g,\underline{g})\in G\times G^j = G^{j+1}$. This says exactly that E_j is a free left $\mathbf{Z}[G]$-module with basis given by elements of G^{j+1} whose initial component is 1. Thus, to define an isomorphism of $\mathbf{Z}[G]$-modules $\phi_j : F_j \simeq E_j$ for $j \geq 0$ it suffices to define a bijection between their indicated bases. We shall use the bijection

$$\phi_j : (g_1,\dots,g_j) \mapsto (1, g_1, g_2g_2, \dots, g_1g_2\cdots g_j).$$

(This is bijective since G is a group!)

Under the isomorphisms ϕ_i for $i \geq 0$, for $j \geq 1$ we claim that the $\mathbf{Z}[G]$-linear map $\mathrm{d}_j : F_j \to F_{j-1}$ goes over to the $\mathbf{Z}[G]$-linear map $\delta_j : E_j \to E_{j-1}$ defined on the standard $\mathbf{Z}$-basis elements $(g_0,\dots,g_j)$ by

$$\delta_j((g_0,\dots,g_j)) = \sum_{i=0}^{j}(-1)^i(g_0,\dots,\widehat{g}_i,\dots,g_j) \tag{2.2}$$

(where, as usual, $\widehat{x}$ means "omit x"). Indeed, δ_j as just defined is visibly $\mathbf{Z}[G]$-linear (check!), so its agreement with d_j reduces to a comparison on the $\mathbf{Z}[G]$-bases. More specifically, for $j \geq 1$ the element $(g_1,\dots,g_j)\in F_j$ is carried by ϕ_j over to the element

$$(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_j) \in E_j$$

that in turn is carried by δ_j over to

$$(g_1, g_1g_2,\dots,g_1g_2\cdots g_j) + \sum_{i=1}^{j-1}(-1)^i(1,\dots,(g_1\cdots g_i)^\wedge,\dots,g_1g_2\cdots g_j) + (-1)^j(1,g_1,g_1g_2,\dots,g_1g_2\cdots g_{j-1}).$$

This is exactly

$$[g_1]\cdot\phi_j(g_2,\dots,g_j) + \sum_{i=1}^{j-1}(-1)^i\phi_{j-1}(g_1,\dots,g_ig_{i+1},\dots,g_j) + (-1)^j\phi_{j-1}(g_1,\dots,g_{j-1}),$$

which is $\phi_{j-1}(\Delta_j(g_1,\dots,g_j))$ and so proves that $\delta_j\circ\phi_j = \phi_{j-1}\circ\mathrm{d}_j$ for all $j\geq 1$.

To prove $\mathrm{d}^2 = 0$ in degrees > 2 it is the same to prove $\delta^2 = 0$ in degrees > 2. We can just as readily prove $\delta^2 = 0$ in degrees ≥ 2: it is a trivial "alternating sum cancellation" computation with the clean formula (2.2) to check (do it!) that $\delta_{j-1}\circ\delta_j = 0$ for all $j \geq 2$. ■

The $\mathbf{Z}$-linear (not $\mathbf{Z}[G]$-linear!) homotopy operators $h_j : F_j \to F_{j+1}$ for $j \geq -1$ satisfying $h_{j-1}\circ\mathrm{d}_j + \mathrm{d}_{j+1}\circ h_j = 1$ for $j \geq 0$ shall be defined by their values on the $\mathbf{Z}$-basis (not $\mathbf{Z}[G]$-basis!) elements $[g_0](g_1,\dots,g_j)$ via the concatenation operation:

$$h_j([g_0](g_1,\dots,g_j)) = (g_0,g_1,\dots,g_j).$$

Since the meaning of this expression may be unclear for $j \leq 0$, we explicitly define $h_{-1} : \mathbf{Z} \to \mathbf{Z}[G] = F_0$ by $1 \mapsto (\emptyset) = [1]$ and $h_0 : \mathbf{Z}[G] \to F_1$ by $[g] \mapsto (g) \in F_1$ for $g \in G$.

Proposition 2.2. *For all $j \geq 0$, $h_{j-1} \circ \mathrm{d}_j + \mathrm{d}_{j+1} \circ h_j = \mathrm{id}_{F_j}$.*

Proof. We make a direct computation for $j = 0$ (since we have to use h_{-1}), and then for $j \geq 1$ we carry out a uniform computation using the E_j's rather than the F_j's (since the h_j's are only $\mathbf{Z}$-linear rather than $\mathbf{Z}[G]$-linear, and the E's have a "clean" $\mathbf{Z}$-basis whereas the F's only have a "clean" $\mathbf{Z}[G]$-basis).

For $j = 0$, the endomorphism $h\mathrm{d} + \mathrm{d}h$ of $F_0 = \mathbf{Z}[G]$ carries $[g]$ to

$$h_{-1}(\mathrm{d}_0([g])) + \mathrm{d}_1(h_0([g])) = h_{-1}(1) + \mathrm{d}_1((g)) = [1] + ([g] - [1]) = [g].$$

To handle $j > 0$, we first compute how our isomorphisms $\phi_i : F_i \simeq E_i$ for $i \geq 0$ transfer $h_i : F_i \to F_{i+1}$ into a map $h'_i : E_i \to E_{i+1}$ for $i \geq 0$. We claim that

$$h'_i(g_0, \ldots, g_i) = (1, g_0, \ldots, g_i).$$

Since the isomorphism $F_0 \simeq E_0$ is the identity endomorphism of $\mathbf{Z}[G]$ whereas the isomorphism

$$F_1 = \bigoplus_{g \in G} \mathbf{Z}[G](g) \simeq \mathbf{Z}[G \times G] = E_1$$

is $[g'] \cdot (g) \mapsto [g'] \cdot (1, g) = (g', g'g)$, clearly $h'_0 : E_0 \to E_1$ is $(g) \mapsto (1, g)$, as desired. For $i > 0$ we likewise have that the $\mathbf{Z}$-basis elements $[g_0] \cdot (g_1, \ldots, g_i) \in F_i$ correspond under ϕ_i to

$$[g_0]\phi_i(g_1, \ldots, g_i) = (g_0, g_0g_1, \ldots, g_0g_1 \cdots g_i)$$

whereas $h_i([g_0](g_1, \ldots, g_i)) = (g_0, g_1, \ldots, g_i)$ corresponds under ϕ_i to $(1, g_0, g_0g_1, \ldots, g_0 \cdots g_i)$. Thus,

$$h'_i(g_0, g_0g_1, \ldots, g_0g_1 \cdots g_i) = (1, g_0, g_0g_1, \ldots, g_0 \cdots g_i)$$

for all $g_0, \ldots, g_i \in G$. Since G is a group, this establishes the asserted formula for $h'_i : E_i \to E_{i+1}$ for all $i \geq 0$.

It now suffices to show that $h'_{j-1} \circ \delta_j + \delta_{j+1} \circ h_j = \mathrm{id}_{E_j}$ for all $j \geq 1$. This is a simple computation: the $\mathbf{Z}$-basis element $(g_0, \ldots, g_j)$ is carried to

$$\sum_{i=0}^{j} (-1)^i h'_{j-1}(g_0, \ldots, \widehat{g}_i, \ldots, g_j) + \delta_{j+1}(1, g_0, \ldots, g_j),$$

which is equal to

$$\sum_{i=0}^{j} (-1)^i (1, g_0, \ldots, \widehat{g}_i, \ldots, g_j) + (g_0, \ldots, g_j) + \sum_{i=1}^{j+1} (-1)^i (1, g_0, \ldots, \widehat{g}_{i-1}, \ldots, g_j).$$

Reindexing the final sum by replacing i with $i - 1$ makes the two sums cancel termwise, so the only term that survives is $(g_0, \ldots, g_j)$, as desired. ■

3. Computations

Now we apply the bar resolution to compute group cohomology. The complex $\mathrm{Hom}_{\mathbf{Z}[G]}(F_\bullet, M)$ has jth term

$$\mathrm{Hom}_{\mathbf{Z}[G]}(F_j, M) = \mathrm{Map}(G^j, M)$$

identified with the set of M-valued functions $f : G^j \to M$ of ordered j-tuples in G. We write $[f] : F_j \to M$ to denote the $\mathbf{Z}[G]$-linear map corresponding to the function $f : G^j \to M$. For $j \geq 1$, the differential

$$\mathrm{d}_{j,M} : \mathrm{Map}(G^j, M) = \mathrm{Hom}_{\mathbf{Z}[G]}(F_j, M) \to \mathrm{Hom}_{\mathbf{Z}[G]}(F_{j+1}, M) = \mathrm{Map}(G^{j+1}, M)$$

carries a function $f : G^j \to M$ to the function

$$(g_1, \dots, g_{j+1}) \mapsto [f](\mathrm{d}_{j+1}((g_1 \dots, g_{j+1}))) = [f](\Delta_{j+1}(g_1, \dots, g_{j+1})).$$

Using the $\mathbf{Z}[G]$-linearity of $[f]$, we obtain

$$(\mathrm{d}f)(g_1, \dots, g_{j+1}) = g_1 \cdot f(g_2, \dots, g_{j+1}) + \sum_{i=0}^{j-1} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{j+1}) + (-1)^j f(g_1, \dots, g_j),$$

where the first term on the right side uses the G-action on M. For example, if $j = 1$ then we have

$$(\mathrm{d}f)(g, g') = g \cdot f(g') - f(gg') + f(g)$$

whereas if $j = 2$ we have

$$(\mathrm{d}f)(g_1, g_2, g_3) = g_1 \cdot f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2).$$

With appropriate conventions regarding the empty set, this formula for $\mathrm{d}f \in \mathrm{Map}(G^{j+1}, M)$ also works for $j = 0$. Explicitly, for an element $f \in \mathrm{Hom}_{\mathbf{Z}[G]}(F_0, M) = \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[G],) = M = \mathrm{Map}(G^0, M)$ corresponding to $m \in M$,

(3.1) $$(\mathrm{d}f)(g) = g \cdot m - m$$

in $\mathrm{Map}(G, M) = \mathrm{Hom}_{\mathbf{Z}[G]}(F_1, M)$.

By definition, for $j > 0$ we have $\mathrm{H}^j(G, M) = Z^j(G, M)/B^j(G, M)$ where

$$Z^j(G, M) = \ker(\mathrm{d}_{j,M}) = \{f : G^j \to M \mid \mathrm{d}f : G^{j+1} \to M \text{ vanishes}\},$$

$$B^j(G, M) = \mathrm{im}(\mathrm{d}_{j-1,M}) = \{f : G^j \to M \mid f = \mathrm{d}h \text{ for some } h : G^{j-1} \to M\}$$

(with the understanding for $j = 1$ that $h : G^0 \to M$ means just an element of M). Functions in $Z^j(G, M)$ are called *j-cocycles* on G valued in M, and functions in $B^j(G, M)$ are called *j-coboundaries* on G valued in M. Elements of $Z^j(G, M)$ which represent the same class in $\mathrm{H}^j(G, M)$ are called *cohomologous*.

Example 3.1. The 1-cocycles and 1-coboundaries are respectively

$$Z^1(G, M) = \{f : G \to M \mid f(gg') = g \cdot f(g') + f(g)\}, \quad B^1(G, M) = \{g \mapsto gm - m \mid m \in M\}.$$

Classically elements of $Z^1(G, M)$ were called *crossed homomorphisms* (as they are almost like homomorphisms, except for the intervention of the G-action in a slightly asymmetric manner).

A 2-cocycle on G with values in M is a function $f : G^2 \to M$ that satisfies

$$g \cdot f(g', g'') + f(g, g'g'') = f(gg', g'') + f(g, g').$$

Classically these were called *factor systems*. Such 2-variable functions first arose early in the 20th century (long before the development of group cohomology!) in the attempts by Noether, Brauer, Dickson, and others to construct and classify certain kinds of finite-dimensional associative algebras over fields (with G a Galois group). The condition of factor systems being cohomologous was known in those days, as it is exactly the condition that the associative algebras arising from two factor systems are abstractly isomorphic.

We finish this handout by computing the connecting map $\partial_j : \mathrm{H}^j(G, M'') \to \mathrm{H}^{j+1}(G, M')$ arising from a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of G-modules, especially the case $j = 0$ (which comes up *all the time* in examples). Let's first work out $j = 0$ by hand, and then treat $j > 0$ in a uniform manner. This is all just an unwinding of the general snake lemma method by which δ-functors are built via acyclic resolutions.

The case $j = 0$. For $m'' \in M''^G$, we can pick a lift $m \in M$, but usually it is not G-invariant. However, $gm - m$ has image $gm'' - m'' = 0$ in M'', so it is valued in $\ker(M \to M'') = M'$. That is, we get a function $G \to M'$ via $g \mapsto gm - m$. This M'-valued function on G is *not a* 1*-coboundary*! When viewed with values in the module M it is a 1-cobounary, but from the viewpoint of M' is may not be a 1-coboundary (since $m \in M$, and usually $m \notin M'$).

The M'-valued function $g \mapsto gm - m$ *is* a 1-cocycle. Indeed, this can be verified by hand, but it is slicker to observe that it suffices to check the 1-cocycle identity in the $\mathbf{Z}[G]$-module M' after applying the $\mathbf{Z}[G]$-linear *injection* into M, where this function *is* a 1-coboundary (and hence a 1-cocycle, so it satisfies the 1-cocycle identity). We claim that the class in $\mathrm{H}^1(G, M')$ of the 1-cocycle $g \mapsto gm - m \in M'$ is $\partial_0(m'')$.

[In particular, this cohomology class only depends on m'', which is to say that changing m has the effect of changing our 1-cocycle by a 1-coboundary. But this latter property is clear by hand: to change m lifting m'' amounts to replacing m with $m + m'$ for $m' \in M'$, in which case the function $g \mapsto gm - m$ changes by adding the function $g \mapsto gm' - m'$ that is visibly in $B^1(G, M')$, not just in $Z^1(G, M')$.]

To verify our assertion concerning $\partial_0(m'')$, we note that $\mathrm{Map}(G^0, M) \to \mathrm{Map}(G^0, M'')$ carries m to m'', so the snake lemma method computes $\partial_0(m'')$ by pushing $m \in \mathrm{Map}(G^0, M)$ into $\mathrm{Map}(G^1, M)$ via the differential d_0. By (2.2), this gives the function $f : G^1 \to M$ defined by $g \mapsto gm - m$. The snake method ensures (as we have already noted by hand, using that $m'' \in M''^G$) that this element of $\mathrm{Map}(G^1, M)$ lies in $\mathrm{Map}(G^1, M')$, and as such must be in $Z^1(G, M')$ and represents $\partial_0(m'') \in \mathrm{H}^1(G, M')$. This completes the proof of the correctness of our description of $\partial_0(m'')$ via 1-cocycles on G valued in M'.

The case $j \geq 1$. Fix $j \geq 1$ and pick a j-cocycle $f : G^j \to M''$ representing a class $\xi \in \mathrm{H}^j(G, M'')$. To compute its image in $\mathrm{H}^{j+1}(G, M')$ under the connecting map, the snake method goes as follows. The natural map $\mathrm{Map}(G^j, M) \to \mathrm{Map}(G^j, M'')$ is surjective, and more specifically f lifts to a map $\widetilde{f} : G^j \to M$ where $\widetilde{f}(g_1, \ldots, g_j)$ is an arbitrary choice of lift into M of $f(g_1, \ldots, g_j) \in M''$. Applying the differential $\mathrm{d}_{j,M} : \mathrm{Map}(G^j, M) \to \mathrm{Map}(G^{j+1}, M)$, we get the function

$$(g_1, \ldots, g_{j+1}) \mapsto g_1 \cdot \widetilde{f}(g_2, \ldots, g_{j+1}) + \sum_{i=1}^{j} (-1)^i \widetilde{f}(g_1, \ldots, g_i g_{i+1}, \ldots, g_{j+1}) + (-1)^{j+1} \widetilde{f}(g_1, \ldots, g_j).$$

The general theory ensures many things:

(i) this function is necessarily valued in M' (because $\mathrm{d}_{j,M} f = 0$!),

(ii) the resulting element of $\mathrm{Map}(G^{j+1}, M')$ lies in $Z^{j+1}(G, M')$ (because $\mathrm{d}_{j+1,M} \circ \mathrm{d}_{j,M} \widetilde{f} = 0$),

(iii) the class of this $(j+1)$-cocycle in $\mathrm{H}^{j+1}(G, M')$ represents $\partial_j(\xi)$ (so it is independent of the original representative j-cocycle f for ξ).