Math 210B. Annihilation of cohomology of finite groups

1. Motivation

An important fact in the cohomology theory of finite groups is that if G is a finite group then $\mathrm{H}^n(G, M)$ is killed by $\#G$ for any $n > 0$ and any G-module M. This is not at all obvious from the abstract definition (via injective resolutions, say), nor even from the viewpoint of the explicit "bar resolution" method of computing group cohomology. In this handout we present the proof of this fundamental fact as a marvelous concrete application of the abstract fact that erasable δ-functors are universal (Grothendieck's conceptualization of the "degree-shifting" method that had long been used in earlier versions of cohomology theories throughout algebra and topology).

Before we turn to the proof, we record a remarkable consequence:

Theorem 1.1. *If G is finite and M is a G-module that is finitely generated as an abelian group then $\mathrm{H}^n(G, M)$ is finite for all $n > 0$.*

Proof. From the bar resolution, we see that $\mathrm{H}^n(G, M)$ is finitely generated as an abelian group, as the terms of the bar resolution complex are the abelian groups $\mathrm{Map}(G^j, M)$ which are visibly finitely generated (as each G^j is a finite set). Hence, since it is also killed by $\#G$, it must be finite! ■

Actually, the finiteness of such higher cohomologies can be seen in an entirely different way, at the cost of losing the precise information of being annihilated by the specific integer $\#G$ (which is very useful in some applications). Namely, since the cohomologies are at least finitely generated as abelian groups (due to the bar resolution argument explained above), to prove finiteness it is equivalent to show that the associated $\mathbf{Q}$-vector spaces $\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{H}^n(G, M)$ vanish for $n > 0$. Consider the natural map

$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{H}^n(G, M) \to \mathrm{H}^n(G, M_{\mathbf{Q}})$$

where $M_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} M$ with its evident G-action. I claim that this map is an isomorphism. This can be seen by compatibly computing both sides via the bar resolution and using the exactness of $\mathbf{Q} \otimes_{\mathbf{Z}} (\cdot)$ (and it can also be proved by a universal δ-functor argument which we leave to the interested reader).

Thus, the desired vanishing is reduced to the case of higher G-cohomology of $\mathbf{Q}[G]$-modules. Since the forgetful functor from $\mathbf{Q}[G]$-modules to $\mathbf{Z}[G]$-modules carries injectives to injectives (as $\mathrm{Hom}_{\mathbf{Z}[G]}(N, I) = \mathrm{Hom}_{\mathbf{Q}[G]}(N_{\mathbf{Q}}, I)$ is exact in N for an injective $\mathbf{Q}[G]$-module I), it follows that the restriction of $\mathrm{H}^\bullet(G, \cdot)$ to the category of $\mathbf{Q}[G]$-modules is erasable (!) and thus must coincide with the derived functor of its degree-0 part. That is, the $\mathbf{Z}[G]$-module cohomology δ-functor on the category of $\mathbf{Q}[G]$-modules is the derived functor of the functor of G-invariants on the category of $\mathbf{Q}[G]$-modules. (Think about that: it is not a tautology!) But this latter functor is *exact* on the category of $\mathbf{Q}[G]$-modules since we can use the averaging method with $\mathbf{Q}$-coefficients (as $\#G \in \mathbf{Q}^\times$) to *functorially* split off the G-invariants as a direct summand.

2. Restriction and corestriction

The key to proving that higher G-cohomology is killed by $\#G$ when G is finite is to develop general cohomological operations called *restriction* and *corestriction* relating G-cohomology and G'-cohomology for any finite-index subgroup $G' \subset G$ for *any* group G (not necessarily finite). Taking $G' = 1$ when G is finite will then provide what we need.

First we discuss restriction, as this is easier to understand. Let G be any group, and G' a subgroup (not necessarily of finite index). For any G-module M, we have $M^G \to M^{G'}$. The forgetful functor from $\mathrm{Mod}(\mathbf{Z}[G])$ to $\mathrm{Mod}(\mathbf{Z}[G'])$ is exact, so the functor $M \rightsquigarrow \mathrm{H}^\bullet(G', M)$ is a δ-functor on the category of G-modules. By the universality of erasable δ-functors (such as the derived functor $\mathrm{H}^\bullet(G, \cdot)$), it follows that the natural transformation $M^G \to M^{G'}$ between left-exact functors on the category of G-modules uniquely extends to a natural transformation

$$\mathrm{Res} : \mathrm{H}^\bullet(G, M) \to \mathrm{H}^\bullet(G', M)$$

between δ-functors on the category of G-modules. This is called *restriction*. (These maps are generally *not* injective in positive degrees, even though it is injective in degree 0.)

Restriction in group cohomology has a natural interpretation at the level of the bar resolution complex, also explaining the name "restriction":

Proposition 2.1. *Let M be a G-module, and consider the bar resolutions $C^\bullet(G, M) :=$ $\mathrm{Map}(G^\bullet, M)$ and $C^\bullet(G', M) := \mathrm{Map}(G'^\bullet, M)$. The natural maps $C^\bullet(G, M) \to C^\bullet(G', M)$ corresponding to restriction of functions $f : G^j \to M$ to the subset $G'^j \subset G^j$ is a map of complexes, and the induced maps*

$$\rho_M^\bullet : \mathrm{H}^\bullet(G, M) \to \mathrm{H}^\bullet(G', M)$$

between homologies is the δ-functorial restriction morphism.

Proof. It is easy to check that this is a map of complexes, so the problem is to show that we recover the restriction maps in group cohomology. Since a map between δ-functors (with left-exact 0th term) is uniquely determined by its effect in degree 0 when the source δ-functor is universal (e.g., erasable), and the map built from the bar resolutions in degree 0 is visibly the natural inclusion $M^G \hookrightarrow M^{G'}$, it suffices to show that our construction with the bar resolution is a map of δ-functors.

Consider a short exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules. This is also short exact as G'-modules, and we seek to show the commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{H}^n(G, M'') & \xrightarrow{\delta} & \mathrm{H}^{n+1}(G, M') \\ \downarrow{\scriptstyle \rho^n_{M''}} & & \downarrow{\scriptstyle \rho^{n+1}_{M'}} \\ \mathrm{H}^n(G', M'') & \xrightarrow[\delta]{} & \mathrm{H}^{n+1}(G', M') \end{array}$$

for all $n \geq 0$. But the δ-functor structure can be computed via the bar resolution by applying the snake lemma, and by inspecting how that goes in the language of cocycles and coboundaries (as functions on G^j valued in G-modules, and similarly for G') the commutativity of these diagrams is clear. (Check it!) ∎

Remark 2.2. There is a useful generalization of the restriction morphism $\mathrm{H}^\bullet(G, M) \to \mathrm{H}^\bullet(G', M)$ for G-modules M, based on the observation that injectivity of $G' \to G$ was *never* used above. Namely, if $\phi : G' \to G$ is *any* group homomorphism then we get a "forgetful" functor $\mathrm{Mod}(\mathbf{Z}[G]) \to \mathrm{Mod}(\mathbf{Z}[G'])$ from the category of G-modules to the category of G'-modules by converting any G-module M into a G'-module by making G' act through ϕ. That is, for $g' \in G'$ we define $g'.m = \phi(g')m$ for $m \in M$. We still have a "forgetful" map $M^G \to M^{G'}$ for any G-module M, generally just an inclusion though it is an equality when $G' \to G$ is surjective.

Since the forgetful functor from G-modules to G'-modules is exact, $M \rightsquigarrow \mathrm{H}^\bullet(G', M)$ on the category of G-modules is a δ-functor (generally not erasable, though below we will see that it is erasable when $G' \to G$ is injective). By the universality of derived functors, it follows exactly as above that the natural map $M^G \to M^{G'}$ uniquely extends to a morphism of δ-functors

$$\mathrm{Res}^\bullet_\phi : \mathrm{H}^\bullet(G, M) \to \mathrm{H}^\bullet(G', M)$$

for G-modules M. This δ-functor map respects composition in the homomorphism $G' \to G$ since we can check it in degree 0 (by a universal δ-functor argument). In the special case when $G' \to G$ is surjective (say with kernel H) one usually calls $\mathrm{Res}^\bullet_\phi$ by the name *inflation*:

$$\mathrm{Inf} : \mathrm{H}^\bullet(G'/H, M) \to \mathrm{H}^\bullet(G', M)$$

for modules M over $G = G'/H$ (extending the evident equality in degree 0).

The proof of Proposition 2.1 carries over verbatim to the case of an arbitrary homomorphism $\phi : G' \to G$ (not necessarily injective), so in general $\mathrm{Res}^\bullet_\phi$ is computed at the level of cochains by composing a cochain $f : G^j \to M$ with $\phi^j : G'^j \to G^j$.

Next, we introduce a map of δ-functors in the opposite direction called *corestriction* when G' has finite index in G. The motivation comes from a construction in degree 0. Assuming G' has finite index in G, if M is a G-module then we can create G-invariant elements of M from G'-invariant elements by a finite averaging process: $M^{G'} \to M^G$ is defined by

$$m \mapsto \sum_{g \in G/G'} gm$$

(where we use coset representations for G/G', the choice of which do not matter since m is G'-invariant). The sum over G/G' makes sense since this quotient set is finite, and the sum is visibly G-invariant: applying the action of $g \in G$ just permutes the terms in the sum.

In order to move to higher degrees, we need to know that G'-cohomology applied to the category of G-modules is universal on the category of G-modules (not G'-modules!). Even better, we claim that this δ-functor is erasable, so it is the derived functor of $M \rightsquigarrow M^{G'}$ on the category of G-modules. To prove the erasability we will use injective G-modules (as there are "enough" of those!). It suffices to show that an injective G-module is injective as a G'-module (as the latter kill higher G'-cohomology). This works without any conditions on the index of G' in G:

Lemma 2.3. *For any group G and subgroup G', if J is an injective G-module then it is also an injective G'-module.*

Proof. The problem is to show that $\mathrm{Hom}_{\mathbf{Z}[G']}(\cdot, J)$ is an exact functor on the category of G'-modules. We will express it as a composition of exact functors. If M' is a $\mathbf{Z}[G']$-module then

$$\mathrm{Hom}_{\mathbf{Z}[G']}(M', J) = \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} M', J)$$

where the tensor product $\mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} M'$ uses the left $\mathbf{Z}[G']$-module structure of J and the right $\mathbf{Z}[G']$-module structure of $\mathbf{Z}[G]$. Since $\mathbf{Z}[G]$ is free as a right $\mathbf{Z}[G']$-module, the functor

$$M' \rightsquigarrow \mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} M'$$

is exact, and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, J)$ is exact since J is G-injective. We conclude that $\mathrm{Hom}_{\mathbf{Z}[G']}(\cdot, J)$ on the category of $\mathbf{Z}[G']$-modules is a composition of exact functors and thus is exact. ■

By the universality of derived functors and our result that the δ-functor of G'-cohomology of G-modules *is* the derived functor of its 0th term on the category of G-modules (and so is erasable), there is a *unique* map of δ-functors

$$\mathrm{Cor} : \mathrm{H}^\bullet(G', M) \to \mathrm{H}^\bullet(G, M)$$

which in degree 0 is $m \mapsto \sum_{g \in G/G'} gm$. We call this δ-functor morphism *corestriction*.

Remark 2.4. It is possible to compute corestriction in terms of the bar resolution. (The general nonsense theory of δ-functors does not guarantee that this should be possible: the mere existence of a homomorphism of δ-functors does not ensure that it can be induced by a homomorphism at the level of a specific resolving complex.) However, it turns out to be rather tricky to formulate this description of corestriction, and even trickier to find a reference which actually gives the formula! (Proof: I once tried it on Google and failed.) One such reference is Lang's book "Cohomology of groups". In fact, even when the appropriate formula is written down, proving it is correct (equivalently, it defines a map of δ-functors) is a bit of a mess. For example, Lang's book on cohomology of groups does not give a proof of the correctness.

Fortunately, such an explicit description is often not necessary: we can frequently just compute generally in degree 0 and then let Grothendieck's δ-functor uniqueness theorem do all of the work to propagate to higher degrees. This is what will happen below.

3. The vanishing proof

Now comes the key point:

Theorem 3.1. *Let G be a group, G' a subgroup of finite index d. For any G-module M, the composite map*

$$\mathrm{H}^\bullet(G, M) \overset{\mathrm{Res}}{\to} \mathrm{H}^\bullet(G', M) \overset{\mathrm{Cor}}{\to} \mathrm{H}^\bullet(G, M)$$

is multiplication by d.

Proof. The problem in question is to compare two endomorphisms of the universal δ-functor $\mathrm{H}^\bullet(G, \cdot)$ on the category of G-modules. By universality it suffices to compare in degree 0, and this case is trivial: if $m \in M^G$ then $\sum_{g \in G/G'} gm = \#(G/G')m = d \cdot m$. ■

Suppose G is finite of size d, so we can take $G' = \{1\}$ in the above theorem. It follows that multiplication by d on $\mathrm{H}^n(G, M)$ factors through maps into and out of $\mathrm{H}^n(1, M)$. For the trivial group, the functor of invariants is *exact* and hence has vanishing *higher* derived functors, so $H^n(1, \cdot) = 0$ for $n > 0$. Thus, if $n > 0$ then multiplication by d on $\mathrm{H}^n(G, M)$ factors through 0 and so vanishes. This is the annihilation result that we wanted to prove.

Remark 3.2. Since it was noted above that corestriction can be described at the level of the bar resolution, it is natural to wonder if the composition of the formulas for corestriction and restriction at the level of the bar resolution is equal to d prior to passing to homologies. I don't remember the corestriction formula, so I haven't checked this (and it might be a mess to do so). In particular, I don't remember offhand if the answer to this question is affirmative or not. But for our purposes this ignorance doesn't matter.