SOLVABLE AND NILPOTENT GROUPS

If $A, B \subseteq G$ are subgroups of a group G, define $[A, B]$ to be the subgroup of G generated by all commutators $\{[a, b] = aba^{-1}b^{-1} \mid a \in A, b \in B\}$. Thus, the elements of $[A, B]$ are finite products of such commutators and their inverses. Since $[a, b]^{-1} = [b, a]$, we have $[A, B] = [B, A]$.

If both A and B are normal subgroups of G then $[A, B]$ is also a normal subgroup. (Clearly, $c[a, b]c^{-1} = [cac^{-1}, cbc^{-1}]$.) Recall that a *characteristic [resp fully invariant]* subgroup of G means a subgroup that maps to itself under all automorphisms [resp. all endomorphisms] of G. It is then obvious that if A, B are characteristic [resp. fully invariant] subgroups of G then so is $[A, B]$.

Define a series of normal subgroups

$$G = G_{(0)} \supseteq G_{(1)} \supseteq G_{(2)} \supseteq \cdots$$

$$G_{(0)} = G, \quad G_{(n+1)} = [G_{(n)}, G_{(n)}].$$

Thus $G_{(n)}/G_{(n+1)}$ is the derived group of $G_{(n)}$, the universal abelian quotient of $G_{(n)}$. The above series of subgroups of G is called the *derived series* of G.

Define another series of normal subgroups of G

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

$$G_0 = G, \quad G_{n+1} = [G, G_n].$$

This second series is called the *lower central series* of G. Clearly, both the $G_{(n)}$ and the G_n are fully invariant subgroups of G.

DEFINITION 1: Group G is *solvable* if $G_{(n)} = \{1\}$ for some n.

DEFINITION 2: Group G is *nilpotent* if $G_n = \{1\}$ for some n.

We will first study solvable groups. But note that an easy induction gives $G_{(n)} \subseteq G_n$, so if G is nilpotent then it is certainly solvable. Also, $G_{(1)} = G_1 = [G, G]$, which is the trivial subgroup $\{1\}$ exactly when G is abelian. So both solvability and nilpotence can be viewed as a kind of upper bound of non-abelianness, iterated commutators of sufficient complexity are trivial, with nilpotent groups lying between the abelian groups and the solvable groups.

PROPOSITION 1: The following are equivalent:
(i) G is solvable with $G_{(n)} = \{1\}$.

(ii) There exists a finite sequence of normal subgroups of G

$$G = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n = \{1\}$$

with each successive quotient A_j/A_{j+1} abelian.
(iii) There exists a finite sequence of subgroups of G

$$G = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n = \{1\}$$

with each B_{j+1} normal in B_j and with each successive quotient B_j/B_{j+1} abelian.

PROOF: (i) $\Rightarrow$ (ii) Take $A_j = G_{(j)}$.
(ii) $\Rightarrow$ (iii) Take $B_j = A_j$.
(ii) $\Rightarrow$ (iii) Since G/B_1 is abelian, we have $G_{(1)} \subset B_1$. Inductively, if $G_{(j)} \subset B_j$ then B_j/B_{j+1} abelian implies $G_{(j+1)} \subset B_{j+1}$. Thus $G_{(n)} = \{1\}$.

PROPOSITION 2(i): Subgroups $H \subset G$ and quotient groups G/K of a solvable group G are solvable.
(ii) If normal subgroup $N \lhd G$ is solvable and if the quotient G/N is solvable then G is solvable.

PROOF: (i) $H_{(n)} \subset G_{(n)}$ for all n. Also, $G_{(n)}$ maps *onto* $(G/K)_{(n)}$ for all n.
(ii) If $(G/N)_{(n)} = \{1\}$ then $G_{(n)} \subset N$. So if $N_{(m)} = \{1\}$ then $G_{(n+m)} \subset N_{(m)} = \{1\}$.

PROPOSITION 3: A *finite* group G is solvable if and only if there exists a finite sequence of subgroups of G

$$G = C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k = \{1\}$$

with each C_{j+1} normal in C_j and with each successive quotient C_j/C_{j+1} finite cyclic of prime order.

PROOF: This is immediate from Proposition 1, along with the fact that any finite abelian group has a composition series with successive quotients cyclic of prime order. One just inserts such a composition series for B_j/B_{j+1} "between" B_j and B_{j+1} in a series of type Proposition 1(iii), say.

It is a very profound theorem of Thompson and Feit that all finite groups of odd order are solvable. This result is acknowledged as being the main first step in the work that eventually culminated in the classification of all finite simple groups.

Every finite group G has a composition series, and the collection of successive quotients is a collection of finite simple groups independent of the choice of composition series. (Jordan-Holder Theorem.) The solvable groups are thus those groups whose simple successive quotients in a composition series are (prime cyclic) abelian groups.

The smallest non-solvable group is the simple group A_5, the alternating group of order 60 inside the symmetric group S_5.

Now we turn to nilpotent groups. First, here are three easy identities involving simple commutators of length 2. Write ${}^g h = ghg^{-1}$. Then

$$[x, y^{-1}] = [{}^{(y^{-1})}x, y]^{-1}, \qquad {}^y[x, z] = [{}^y x,\ {}^y z], \qquad [x, yz] = [x, y]\ {}^y[x, z].$$

We have defined $G_{n+1} = [G, G_n]$. Thus, according to the definition, inductively each G_n is generated by a complicated set of elements. However, it is easy to write down a simpler list of generators of G_n. Namely, define simple commutators of length $n+1 \geq 3$ inductively by

$$[x_1, x_2, \cdots, x_{n+1}] = [x_1, [x_2, \cdots, x_{n+1}]].$$

Then the identities above, and an induction, shows the following.

PROPOSITION 4: For all $n \geq 1$, the subgroup G_n is generated by simple commutators of length $n+1$.

The proof of Proposition 2(i) applied to the series G_n instead of $G_{(n)}$ immediately gives the first statement below. The second statement is equally easy.

PROPOSITION 5: Subgroups $H \subset G$ and quotient groups G/K of a nilpotent group G are nilpotent. The direct product of two nilpotent groups is nilpotent.

However the analogue of Proposition 2(ii) is not true for nilpotent groups. For example, $[S_3, S_3] = A_3$ but also $[S_3, A_3] = A_3$. Here, $A_3 \subset S_3$ is the (cyclic) alternating group inside the symmetric group on three letters.

We will prove below that p-groups are nilpotent for any prime, and then we will prove that all *finite* nilpotent groups are direct products of their (unique, normal) Sylow-p subgroups. So this is a very strong structure theorem for finite nilpotent groups.

The following result is obvious from the definition $G_{n+1} = [G, G_n]$.

PROPOSITION 6: G_n/G_{n+1} is contained in the center of G/G_{n+1}.

For any group G, denote the center of G by $Z(G)$. Define an increasing series of normal subgroups of G, $\{1\} = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$, called the *ascending central series*, by

$$Z_1 = Z(G) \qquad Z_{n+1}/Z_n = Z(G/Z_n).$$

Of course we are using here the correspondence between (normal) subgroups of G/Z_n and (normal) subgroups of G that contain Z_n. The descending and ascending central series are closely related.

PROPOSITION 7(i): Suppose $G_r \subseteq Z_{n-r}$. Then $G_{r-1} \subseteq Z_{n-r+1}$.

(For example, the hypothesis holds if G is nilpotent with $G_n = \{1\}$ and $r = n$. By decreasing induction on r, one then concludes $G = G_0 = Z_n$.)

(ii): Suppose $G_s \subseteq Z_{n-s}$. Then $G_{s+1} \subseteq Z_{n-s-1}$.

(Thus if $Z_n = G$ then the hypothesis holds with $s = 0$. By induction on s, on concludes $G_n = Z_0 = \{1\}$.)

PROOF(i): There is the obvious surjection $G/G_r \twoheadrightarrow G/Z_{n-r}$, so the center of G/G_r maps to the center $Z(G/Z_{n-r}) = Z_{n-r+1}/Z_{n-r}$. By Proposition 6, we get $G_{r-1} \subseteq Z_{n-r+1}$.

PROOF(ii): By hypothesis, G_s maps to $Z_{n-s}/Z_{n-s-1} = Z(G/Z_{n-s-1})$. Thus $G_{s+1} = [G, G_s] \subseteq Z_{n-s-1}$.

By Proposition 7, the descending central series terminates with $\{1\}$ after n steps if and only if the ascending central series terminates with G after n steps. It is a standard result that if G is a non-trivial finite p-group for some prime p, then $Z(G) \neq \{1\}$. Therefore the ascending central series of a p-group G is strictly increasing until it terminates at G after finitely many steps. So we have proved

PROPOSITION 8: Finite p-groups are nilpotent.

Our final goal will be to show that in any finite nilpotent group G, the Sylow-p subgroups are normal. It is then standard that for each prime p there is a unique Sylow-p subgroup, and G is the direct product of its Sylow-p subgroups.

PROPOSITION 9: Suppose $H = H_0$ is a subgroup of a group G. Define $H_{i+1} = N_G(H_i)$, the successive normalizers, for $i \geq 1$. Then $Z_i \subseteq H_i$ for all $i \geq 0$. Consequently, if G is nilpotent with $Z_n = G$ then $H_n = G$, so no proper subgroup $H \subsetneq G$ is its own normalizer.

PROOF: The proof is by induction on i. Trivially $Z_0 = \{1\} \subseteq H_0$. If $z_{i+1} \in Z_{i+1}$ and $h_i \in H_i$ then by definition of the terms in the ascending central series,

$$z_{i+1} h_i z_{i+1}^{-1} h_i^{-1} = z_i \in Z_i.$$

By induction, $z_i \in H_i$, so $z_{i+1} h_i z_{i+1}^{-1} \in H_i$. This means $z_{i+1} \in N_G(H_i) = H_{i+1}$.

We continue the notation of Proposition 9.

PROPOSITION 10: Suppose G is any finite group and $H \subset G$ is a Sylow-p subgroup. Then $H_2 = H_1$. That is, $H_1 = N_G(H)$ is its own normalizer in G. Consequently, if G is nilpotent, Proposition 9 implies $N_G(H)$ cannot be a proper subgroup of G, hence H is normal in G.

PROOF: H is a Sylow-p subgroup of $N_G(H)$. Suppose $g \in G$ normalizes $N_G(H)$. Then $gHg^{-1} \subset N_G(H)$ is also a Sylow-p subgroup. Since any two Sylow-p subgroups of a group are conjugate, there is $k \in N_G(H)$ with $kHk^{-1} = gHg^{-1}$. Therefore $k^{-1}g \in N_G(H)$, hence also $g \in N_G(H)$.