

1. SEPARABLE FIELD EXTENSIONS AND COVERING SPACES

**1.1. Review of covering spaces.** Let  $X$  be a path-connected topological space, and  $x_0 \in X$  a point. The fundamental group  $\pi_1(X, x_0)$  classifying homotopy classes of loops in  $X$  based at  $x_0$  is a key tool in the study of covering spaces  $q : X' \rightarrow X$ , at least when  $X$  is “nice” (which we shall now assume; the precise definition of “nice” is explained in courses on algebraic topology).

The group  $\text{Aut}(X'/X)$  acts faithfully on the fiber  $q^{-1}(x_0)$ , and if  $X'$  is also connected then the effect on a single point  $x'_0 \in X'$  over  $x_0$  determines each element of  $\text{Aut}(X'/X)$ . In particular, if  $X'$  is a connected degree- $d$  covering space then  $\#\text{Aut}(X'/X) \leq d$ . Thus, for a finite-degree *connected* covering space, we see that  $\#\text{Aut}(X'/X) = d$  if and only if this automorphism group acts transitively on the fiber  $q^{-1}(x_0)$ . When the equality (or equivalently, transitivity) holds, we say that  $X'$  is *Galois* over  $X$ . (Note the analogy with the criterion in field theory that for a degree- $d$  finite extension  $L/k$  we always have  $\#\text{Aut}(L/k) \leq d$ , with equality if and only if  $L$  is Galois over  $k$ .) In general, without finite-degree hypotheses, we say  $X'$  is *Galois* over  $X$  when  $\text{Aut}(X'/X)$  acts transitively on the fiber  $q^{-1}(x_0)$ .

A *universal covering space* of  $X$  is a connected covering space  $\tilde{X} \rightarrow X$  such that  $\tilde{X}$  has no nontrivial connected covering spaces; equivalently, every covering space of  $\tilde{X}$  is a disjoint union of copies of  $\tilde{X}$ . Since  $X$  is “nice”, this exists and is unique up to  $X$ -isomorphism. (Note the analogy with a separable closure  $k_s/k$ : a separable algebraic extension with no nontrivial separable algebraic extensions of itself, and it is unique up to  $k$ -isomorphism.) The fundamental group  $\pi_1(X, x_0)$  can be identified with the automorphism group  $\text{Aut}(\tilde{X}/X)$  of a connected universal covering space  $\tilde{X} \rightarrow X$  *equipped with* a choice of base point  $\tilde{x}_0 \in \tilde{X}$ , as follows. Consider the homotopy class of a loop  $\gamma : S^1 \rightarrow X$  based at  $x_0$ . This loop uniquely lifts to a path  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{X}$  that begins at  $\tilde{x}_0$ , and we associate to the homotopy class of  $\gamma$  the unique  $X$ -automorphism of  $\tilde{X}$  that carries  $\tilde{x}_0$  to the point  $\tilde{\gamma}(1)$  in the  $x_0$ -fiber of  $\tilde{X}$ . (One checks that this automorphism only depends on  $\gamma$  up to based homotopy, and that this assignment is homomorphic with respect to the composition law in the fundamental group and composition of automorphisms.)

**1.2. Base point and functoriality.** Since  $\tilde{X}$  has lots of  $X$ -automorphisms in general, the canonicity of  $\tilde{X}$  is only up to  $X$ -isomorphism. However, we can rigidify things: if we *choose* a point  $\tilde{x}_0 \in \tilde{X}$  over  $x_0 \in X$ , as we have seen is implicit in the definition of the isomorphism  $\pi_1(X, x_0) \simeq \text{Aut}(\tilde{X}/X)$  anyway, then we get that the *pair*  $(\tilde{X}, \tilde{x}_0)$  is unique up to *unique* isomorphism. That is, if  $\tilde{X}' \rightarrow X$  is another universal covering space and  $x'_0 \in \tilde{X}'$  is a point over  $x_0 \in X$  then there is a *unique*  $X$ -isomorphism of pointed spaces  $(\tilde{X}, \tilde{x}_0) \simeq (\tilde{X}', x'_0)$ .

The pair  $(\tilde{X}, \tilde{x}_0)$  has a universal mapping property: for *any* connected pointed covering space  $q : (X', x'_0) \rightarrow (X, x_0)$  there is a unique map of pointed spaces  $(\tilde{X}, \tilde{x}_0) \rightarrow (X', x'_0)$  over  $X$ . Indeed, the pullback pointed space

$$(\tilde{X} \times_{X, q} X', (\tilde{x}_0, x'_0))$$

is a covering space of  $\tilde{X}$ , so it is a disjoint union of copies of  $\tilde{X}$ , and the base point  $(\tilde{x}_0, x'_0)$  picks out a preferred such component. That is, we get a canonical identification of  $\tilde{X}$  as a connected component of  $\tilde{X} \times_X X'$ , and composing this with the projection  $\text{pr}_2 : \tilde{X} \times_X X' \rightarrow X'$  provides the desired continuous  $X$ -morphism  $(\tilde{X}, \tilde{x}_0) \rightarrow (X', x'_0)$  and also proves its *uniqueness*.

The non-canonical choice that matters is  $x_0 \in X$ , in terms of which the pair  $(\tilde{X}, \tilde{x}_0)$  is canonical in an even stronger sense: if  $f : (Y, y_0) \rightarrow (X, x_0)$  is an arbitrary continuous map between pointed “nice” topological spaces then there is a *unique* lifting of  $f$  to a continuous map

$$\tilde{f} : (\tilde{Y}, \tilde{y}_0) \rightarrow (\tilde{X}, \tilde{x}_0)$$

between the *pointed* universal covers. Explicitly, to define  $\tilde{f}$  we consider the (possibly disconnected!) pullback covering space  $Y \times_{f, X} \tilde{X}$  of  $Y$  equipped with the base point  $y' = (y_0, \tilde{x}_0)$ . The preceding discussion shows that there is a unique  $Y$ -morphism of pointed connected covering spaces from  $\tilde{Y}$  to the connected component of  $Y \times_X \tilde{X}$  containing  $y'$ , and composition of this  $Y$ -morphism with  $\text{pr}_2 : Y \times_X \tilde{X} \rightarrow \tilde{X}$  then provides the desired continuous map  $\tilde{Y} \rightarrow \tilde{X}$  carrying  $\tilde{y}_0$  to  $\tilde{x}_0$ , and also proves its *uniqueness*.

**1.3. Analogy with field theory.** We regard separable extension fields as analogues of connected covering spaces, with the degree of a covering space of  $X$  corresponding to the degree of a field extension. Moreover, the *pointed* universal covering space  $(\tilde{X}, \tilde{x}_0)$  is the analogue of a separable closure. We have seen above that the pair  $(\tilde{X}, \tilde{x}_0)$  is functorial in the pair  $(X, x_0)$ , so in fact the “choice” that occurs on the topological side is really  $x_0 \in X$ . Hence, the choice of the base point  $x_0 \in X$  plays a role similar to that of a *choice* of separable closure of a field, although in practice the covering space  $\tilde{X} \rightarrow X$  is the geometric object that more closely resembles  $k_s/k$ . (This can be a bit confusing: in some respects  $k_s$  is the analogue of  $x_0 \in X$ , and in other respects it is the analogue of  $\tilde{X} \rightarrow X$ . This dual personality issue is due to the fact that  $\text{Spec}(k_s)$  is a single point, and is clarified when one develops the generalization of Galois theory to higher-dimensional schemes.)

Automorphism groups  $\text{Aut}(X'/X)$  of covering spaces  $X' \rightarrow X$  are analogous to automorphism groups of field extensions. Moreover, in the study of covering spaces in topology, there is a “Galois-like” correspondence: subgroups  $H \subset \text{Aut}(\tilde{X}/X) = \pi_1(X, x_0)$  correspond bijectively to factorizations  $\tilde{X} \rightarrow X' \rightarrow X$  with  $X' \rightarrow X$  a covering space, via the inverse recipes  $H \mapsto X' = H \backslash \tilde{X}$  and  $X' \mapsto H = \text{Aut}(\tilde{X}/X')$ . Moreover, one shows that  $X'$  is Galois over  $X$  if and only if the associated subgroup  $H \subset \pi_1(X, x_0)$  is normal, in which case  $\pi_1(X, x_0)/H$  is identified with  $\text{Aut}(X'/X)$ .

## 2. FUNCTORIALITY AND CONJUGATION

The aim of this section is to explain how the intervention of conjugation ambiguity in the functoriality of absolute Galois groups is similar to the appearance of conjugation in the functoriality of fundamental groups. It also turns out that Galois cohomology, to be discussed later in the course, exhibits many features of topological cohomology, though the full force of the analogy can only be seen when fields are replaced with higher-dimensional algebro-geometric objects. (The most satisfactory explanation for these analogies is to be found in Grothendieck’s work on étale fundamental groups and étale cohomology of schemes, in SGA1 and SGA4 respectively. Needless to say, that lies far beyond the scope of this course.)

Let  $j : k \rightarrow F$  be an extension of fields, and choose separable closures  $k_s/k$  and  $F_s/F$ . Since  $F_s$  is thereby viewed as a separably closed extension of  $k$  (via  $k \rightarrow F \rightarrow F_s$ ), the separable closure  $K$  of  $k$  in  $F_s$  is then a separably closed field. Indeed, any separable monic polynomial  $f \in K[x]$  is also a separable polynomial in  $F_s[x]$ , so  $f$  splits completely in  $F_s[x]$  due to  $F_s$  being separably closed, yet the roots of  $f$  in  $F_s$  are all algebraic over  $k$  (since  $f \in K[x]$  with  $K$  algebraic over  $k$ ), so all such roots would have to lie in  $K$  due to the definition of  $K$ . This shows that  $f$  splits completely in  $K[x]$ , so  $K$  is separably closed. It follows that  $K$  is abstractly  $k$ -isomorphic to  $k_s$ , so

upon choosing such an isomorphism we get a commutative diagram of field extensions

$$\begin{array}{ccc} k_s & \xrightarrow{\tilde{j}} & F_s \\ \uparrow & & \uparrow \\ k & \xrightarrow{j} & F \end{array}$$

Viewing  $F$  as living “over”  $k$ , the data of  $j$  is analogous to a continuous map  $f : Y \rightarrow X$  between (nice) connected topological spaces (where  $Y$  is in the role of  $F$  and  $X$  is in the role of  $k$ ; beware of the direction of the arrows), and the construction of  $\tilde{j}$  is analogous to our construction of a continuous lift  $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$  between universal covering spaces. The picture is this:

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xleftarrow{f} & Y \end{array}$$

Alternatively, if we consider the choice of separable closure as akin to picking a base point, then the specification of  $\tilde{j}$  is similar to choosing base points  $y_0 \in Y$  and  $x_0 \in X$  compatibly with  $f$  (i.e.,  $f(y_0) = x_0$ ), and we have seen above that in such a situation there is a *unique* continuous lift between *pointed* universal covering spaces  $\tilde{f} : (\tilde{Y}, \tilde{y}_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ .

Now we turn to the more important task: to what extent is  $\text{Gal}(k_s/k)$  functorial in  $k$ ? We will see that there is some conjugation ambiguity inherent in this question, and it will have a topological analogue for the problem of the functoriality of  $\pi_1(X, x_0)$  in the topological space  $X$  (without reference to  $x_0$ ). First, observe that for a given field extension  $j : k \rightarrow F$ , the lift  $\tilde{j} : k_s \rightarrow F_s$  is not at all unique: we can precompose it with any  $\sigma \in \text{Gal}(k_s/k)$ . But this is the only ambiguity: if  $\tilde{j}' : k_s \rightarrow F_s$  is a lift of  $j$  then necessarily  $\tilde{j}' = \tilde{j} \circ \sigma$  for a (unique)  $\sigma \in \text{Gal}(k_s/k)$ . Indeed, the images  $\tilde{j}'(k_s), \tilde{j}(k_s) \subseteq F_s$  are the *same* subfield, namely the separable closure  $K$  of  $k$  in the extension  $k \rightarrow F \rightarrow F_s$ , so for every  $x \in k_s$  there is a unique  $\sigma(x) \in k_s$  such that  $\tilde{j}'(x) = \tilde{j}(\sigma(x))$  (uniqueness due to the injectivity of  $\tilde{j}$ ). Since  $\tilde{j}$  and  $\tilde{j}'$  are  $k$ -isomorphisms of  $k_s$  onto the same subfield of  $F_s$ , it follows that  $\sigma$  is a  $k$ -automorphism of  $k_s$ , as desired.

Fix one choice of  $\tilde{j}$ . Then any  $\gamma \in \text{Gal}(F_s/F)$  is the identity on  $k$  and so restricts to a  $k$ -automorphism of the separable closure  $K$  of  $k$  in  $F_s$ . But  $\tilde{j} : k_s \simeq K \subset F_s$ , so the restriction of  $\gamma$  to  $K$  defines via  $\tilde{j}$  a  $k$ -automorphism of  $k_s$ . In other words, we have defined a map of groups (easily seen to be a homomorphism)

$$\rho_{\tilde{j}} : \text{Gal}(F_s/F) \rightarrow \text{Gal}(k_s/k)$$

via the condition

$$\tilde{j} \circ \rho_{\tilde{j}}(\gamma) = \gamma \circ \tilde{j}$$

for all  $\gamma \in \text{Gal}(F_s/F)$ . Note the direction in which  $\rho_{\tilde{j}}$  goes. Keeping mind that arrows get “flipped” around under our analogy between field extensions and continuous maps of topological spaces, this is reminiscent of the *covariant* functoriality of  $\pi_1$  relative to a map  $f : (Y, y_0) \rightarrow (X, x_0)$  between pointed (nice) topological spaces. The map  $\rho_{\tilde{j}}$  is analogous to one of two (equivalent) maps: either  $\pi_1(f) : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$  (composing based loops with  $f$ ) or more vividly the map

$$\rho_{\tilde{f}} : \text{Aut}(\tilde{Y}/Y) \rightarrow \text{Aut}(\tilde{X}/X)$$

induced by  $\tilde{f} : (\tilde{Y}, \tilde{y}_0) \rightarrow (\tilde{X}, \tilde{x}_0)$  and characterized by the condition

$$\rho_{\tilde{f}}(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

Explicitly, the composition  $\tilde{f} \circ \gamma : \tilde{Y} \rightarrow \tilde{X}$  over  $f$  carries  $\tilde{y}_0$  to a point  $\tilde{x}_1$  over  $x_0$ , and  $\rho_{\tilde{f}}(\gamma)$  is the unique  $X$ -automorphism of  $\tilde{X}$  that carries  $\tilde{x}_0$  to  $\tilde{x}_1$ , so  $\rho_{\tilde{f}}(\gamma) \circ \tilde{f}, \tilde{f} \circ \gamma : \tilde{Y} \rightarrow \tilde{X}$  are two maps over  $f : Y \rightarrow X$  that carry  $\tilde{y}$  to the same point, and hence they must coincide. (The reader may check that  $\rho_{\tilde{f}} = \pi_1(f)$  under the canonical isomorphism between  $\pi_1(X, x_0)$  and  $\text{Aut}(\tilde{X}/X)$  defined by a choice of  $\tilde{x}_0 \in \tilde{X}$  over  $x_0$ .) Note the close analogy between  $\rho_{\tilde{f}}$  and the field-theoretic construction  $\rho_{\tilde{j}}$ , up to switching the direction of maps (and hence the order of composition).

What happens if we replace  $\tilde{j}$  with  $\tilde{j}' : k_s \rightarrow F_s$  over  $j$ ? We have seen that  $\tilde{j}' = \tilde{j} \circ \sigma$  for a unique  $\sigma \in \text{Gal}(k_s/k)$ , and the reader can readily check that

$$\rho_{\tilde{j}'} = c_{\sigma^{-1}} \circ \rho_{\tilde{j}},$$

where  $c_{\sigma^{-1}}$  is the automorphism of  $\text{Gal}(k_s/k)$  defined by  $\sigma^{-1}$ -conjugation:  $x \mapsto \sigma^{-1}x\sigma$ . Thus, we can say that  $\text{Gal}(k_s/k)$  is *contravariantly* functorial in  $k$  (without reference to  $k_s$ !) *up to conjugation on the target*. The topological analogue is that  $\pi_1(X, x_0)$  is *covariant* in  $X$  (without reference to  $x_0$ !) *up to conjugation on the target*. Indeed, if  $f : Y \rightarrow X$  is continuous but  $x_1 := f(y_0)$  might not equal  $x_0$ , then a choice of path  $\sigma$  in  $X$  linking  $x_0$  to  $x_1$  provides an isomorphism  $\pi_1(X, x_1) \simeq \pi_1(X, x_0)$  whose composition with  $\pi_1(f) : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_1)$  is a homomorphism  $\pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$  that changes by a conjugation if we change the homotopy class of the chosen path between  $x_1$  and  $x_0$ . This is why  $\pi_1(X, x_0)$  is only covariant in  $X$  up to conjugation on the target.

*Example 2.1.* A very interesting example is  $F = k$  (with  $j$  the identity) but  $F_s$  a separable closure of  $F = k$  unrelated to  $k_s$ ! That is, we consider two separable closures  $k_s$  and  $k'_s$  of  $k$ . In this case if we choose a  $k$ -isomorphism  $\tilde{j} : k_s \simeq k'_s$  then  $\rho_{\tilde{j}} : \text{Gal}(k'_s/k) \simeq \text{Gal}(k_s/k)$  is the isomorphism defined by  $\gamma \mapsto \tilde{j}^{-1} \circ \gamma \circ \tilde{j}$  (check!), so we recover in this case the formula  $\rho_{\tilde{j} \circ \sigma} = c_{\sigma^{-1}} \circ \rho_{\tilde{j}}$  for  $\sigma \in \text{Gal}(k_s/k)$ . This expresses the fact that  $\text{Gal}(k_s/k)$  is determined by  $k$  only up to isomorphism that has a conjugation ambiguity. The topological analogue is that  $\pi_1(X, x_0)$  is determined by  $X$  only up to isomorphism that has a conjugation ambiguity.

A striking analogy is seen with homology (or cohomology). By passing to the abelianized fundamental group we kill the effect of conjugation and thereby recover the covariant functoriality in  $X$  of  $H_1(X, \mathbf{Z})$ , and so likewise the contravariant functoriality in  $X$  of  $H^1(X, \mathbf{Z})$ . In a similar spirit, we will see later that *Galois cohomology*  $H^j(\text{Gal}(k_s/k), \cdot)$  is actually functorial in  $k$  alone, without reference to  $k_s$ . One might wonder why the correct analogue of  $\text{Gal}(k_s/k)$ -cohomology isn't group cohomology of  $\pi_1(X, x_0)$  rather than topological (co)homology of  $X$ . The "justification" is best understood within the context of higher-dimensional algebraic geometry (via Grothendieck's brilliant idea of the étale topology on the category of schemes, which puts Galois cohomology into a broader geometric framework).