Extensions and $H^2(G, A)$

To say that a group $E$ is an extension of group $G$ by group $A$ means there is an exact sequence as in the top row of the diagram below. Two extensions are equivalent if there is an isomorphism $\phi$ so that the diagram commutes.

\[
\begin{array}{cccccc}
1 & \rightarrow & A & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
\| & & \downarrow \phi & & \| & & \\
1 & \rightarrow & A & \rightarrow & E' & \rightarrow & G & \rightarrow & 1 \\
\end{array}
\]

Assume $A$ is abelian. An extension produces a $G$-module structure on $A$, by $\hat{x}a = \hat{x}a\hat{x}^{-1}$, where $\hat{x} \in E$ is any lifting of $x \in G$. The conjugation is well-defined because $A$ is abelian and the only other $\hat{x}$'s would be $b\hat{x}$ with $b \in A$. Equivalent extensions induce the same actions, since a map $\phi$ defining an equivalence is assumed to satisfy $\phi|_A = Id$, and it is assumed $\phi$ induces the identity on $G$.

(When $A$ is not abelian, all you get out of this is a homomorphism $G \rightarrow Out(A)$, the outer automorphism group of $A$, rather than a $G$-action $G \rightarrow Aut(A)$).

Even with the action or outer action of $G$ on $A$ fixed, classifying extensions up to equivalence is rather far from classifying all such $E$ up to isomorphism of groups. An isomorphism could discombobulate the $A$'s and may not give rise to any map at all between exact sequences. Nonetheless, classifying extensions is considered important.

**THEOREM:** Equivalence classes of extensions of $G$ by an abelian group $A$, inducing a fixed $G$-module structure on $A$, are in bijective correspondence with elements of the cohomology group $H^2(G, A)$.

We will leisurely prove this theorem in the next couple of pages.

Choose a set $\{f(x) \in E \mid x \in G\}$ of inverse images of all elements of $G$. All elements of $E$ can be uniquely written $af(x)$, for some $a \in A, x \in G$. One could allow $f(1) \neq 1 \in A$, but the formulas below then get a bit ugly. Therefore, by a set of representatives we will always mean such an $f$ with $f(1) = 1$. Note $^*b = f(x)b(f(x))^{-1}$ doesn’t depend on the choice of $f$. Note $f$ is a group homomorphism splitting of the extension sequence if and only if $f(x)f(y)f(xy)^{-1} = 1$ for all $x, y \in G$.

Let’s compute general products in $E$,

\[(af(x))(bf(y)) = a(^*b)(f(x)f(y)f(xy)^{-1})f(xy).\]

The function $u_f(x, y) = f(x)f(y)f(xy)^{-1} \in A$ looks important. The product formula shows that the group structure on $E$ is determined by the $G$-module structure on $A$ and the function $u_f$, since if we identify $E$ setwise with $A \times G$ via $af(x) \leftrightarrow (a, x)$, we can write

\[(a, x)(b, y) \leftrightarrow (af(x))(bf(y)) = a(^*b)u_f(x, y)f(xy) \leftrightarrow (a(^*b)u_f(x, y), xy).\]

Moreover, if $\phi : E \rightarrow E'$ is an isomorphism of extensions, then $\{\phi f(x)\}$ is a set of representatives for $G$ in $E'$. Since $\phi|_A = Id$, it follows that $u_f = u_{\phi f}$.
Recall that a function \( u : G \times G \to A \) determines a cocycle in \( Z^2(G, A) \), for the free standard \( \mathbb{Z}[G] \)-module resolution of \( \mathbb{Z} \), if one has
\[
ud_3(x, y, z) = xu(y, z) - u(xy, z) + u(x, yz) - u(x, y) = 0 \in A.
\]
Except we are writing \( A \) as an abelian multiplicative group here. It is much more convenient to write the cocycle condition as
\[
xu(y, z)u(x, yz) = u(x, y)u(xy, z).
\]

**Exercise 1.** Given a group \( G \), more convenient to write the cocycle condition as

\[
E_a(x, y, z) = xu(y, z) - u(xy, z) + u(x, yz) - u(x, y) = 0 \in A.
\]

Recall that a function \( u : G \times G \to A \) determines a cocycle in \( Z^2(G, A) \), for the free standard \( \mathbb{Z}[G] \)-module resolution of \( \mathbb{Z} \), if one has
\[
ud_3(x, y, z) = xu(y, z) - u(xy, z) + u(x, yz) - u(x, y) = 0 \in A.
\]

At this point, if we have an extension \( 1 \to A \to E \to G \to 1 \) and a set of representatives \( \{f(x)\} \subset E \), then since \( E \) is associative we have a cocycle \( u_f(x, y) = f(x)f(y)f(xy)^{-1} \in A \). Clearly \( u_f(1, 1) = 1 \), since \( f(1) = 1 \).

Because of the explicit product formulas, we see that the extensions \( E \) and \( E_{u_f} = A \times_{u_f} G \) are equivalent, via the isomorphism \( \phi(a f(x)) = (a, x) \). Note \( \phi(a) = (a, 1) = i(a) \in E_u \). The set of representatives \( \{f(x)\} \subset E \) maps to the set of representatives \( \{(1, x)\} \subset E_{u_f} \).

We have worked with (certain) 2-cocycles, now we bring in (certain) 2-coboundaries. Beginning with an extension \( E \) and a set of representatives \( \{f(x)\} \), what are other possible sets of representatives? Obviously just sets \( \{b(x)f(x)\} \), where \( b : G \to A \) is any function with \( b(1) = 1 \). In the standard resolution, \( b \) extends to a \( G \)-map \( F_1 \to A \), with coboundary \( \delta(b) : F_2 \to A \) determined by
\[
\delta(b)(x, y) = bd(x, y) = xb(y) - b(xy) + b(x),
\]
in additive notation. Multiplicatively this can be written
\[
\delta(b)(x, y) = xb(y)b(xy)^{-1}.
\]

Note that \( b(1) = 1 \) implies \( \delta(b)(1, 1) = b(1) = 1 \).
Exercise 4. Show that \( u(bf)(x, y) = \delta(b)u_f(x, y) \). [Hint: \( f(x)b(y) = \ast b(y)f(x) \).]

(Nota that an immediate consequence of Exercise 4 is that the cocycles with \( b \) and \( bf \) determine the same cocycle, \( u_f = u_{bf} \), if and only if \( b \in Z^1(G, A) \), the 1-cocycles, which is exactly the condition that \( b(x) = (b(x), x) \) defines a group homomorphism section of the semidirect product \( A \times_v G \to G \). But this is peripheral to our present discussion.)

Exercise 4 is easy, but has serious consequences. Beginning with an extension \( E \), choose a set of representatives \( f \) and construct the cocycle \( u_f \). Exercise 4 shows that the cohomology class \([u_f] \in H^2(G, A)\) is independent of the choice of set of representatives. So we can call this class \([u(E)] \in H^2(G, A)\). We also observed above that if \( \phi : E \to E' \) is an isomorphism of extensions then \( u_f = u_{\phi f} \), hence \([u(E)] = [u(E')]\). So we have a well-defined map from equivalence classes of extensions (inducing a given \( G \)-module structure on \( A \)) to \( H^2(G, A) \).

We want to prove this correspondence is bijective. But here we must pay a little price for our decision to only consider cocycles \( u \) with \( u(1, 1) = 1 \) and coboundaries \( db \) with \( b(1) = 1 \). Here, the arguments \((1, 1)\) and \((1)\) refer to basis elements in \( F_2 \) and \( F_1 \) of the standard resolution, and \( 1 \in A \) is the identity.

We have \( H^2(G, A) = \frac{Z^2(G, A)}{B^2(G, A)} \), cocycles mod coboundaries. We now go back to additive notation in a general abelian \( G \)-module \( A \). Let \( Z^2_0(G, A) \subset Z^2(G, A) \) denote those cocycles with \( u(1, 1) = 0 \) and let \( B^2_0(G, A) \subset B^2(G, A) \) denote cocycles mod coboundaries \( db \), where \( b(1) = 0 \). We will prove the following claim after finishing the proof of the THEOREM.

CLAIM: \( \frac{Z^2_0(G, A)}{B^2_0(G, A)} \to \frac{Z^2(G, A)}{B^2(G, A)} \) is an isomorphism.

In Exercises 1 and 2, beginning with any cocycle \( u \) with \( u(1, 1) = 1 \), we constructed an extension \( E_u = A \times_v u G \). In Exercise 3 it was verified that \( E_u \) induces the given \( G \)-module structure \( v \) on \( A \), and also that \( u_f = u \) for the natural factor set \( \{f(x)\} = \{(1, x)\} \subset E_u \). Thus, by the claim, the map from equivalence classes of extensions to \( H^2(G, A) \) is surjective. But we also conclude the map is injective. If extension \( E \) yields cocycle \( u \) with one choice of representatives \( f \), then \( E \) and \( E_u \) are equivalent. But replacing \( f \) by \( bf \), where \( b(1) = 1 \), doesn’t change \( E \), and now we see by Exercise 4 that \( E \) is equivalent to \( E_{\phi(b)u} \). In other words, again by the claim, the association \( u \mapsto E_u \) determines a well-defined map from \( H^2(G, A) \) to equivalence classes of extensions, inverse to the association \( E \mapsto [u(E)] \). This proves the THEOREM.

PROOF OF CLAIM: Given any cocycle \( v \in Z^2 \) and a 1-cochain \( c \), note

\[(v + \delta c)(1, 1) = v(1, 1) + cd(1, 1) = v(1, 1) + c(1),\]

since \( d(1, 1) = (1) \) in the standard resolution. But \( c(1) \) is arbitrary, so we can change any cocycle by a coboundary and get \( v + \delta c \in Z^2_0 \). This proves the map in the claim is surjective. But also, if \( u \in Z^2_0 \) and if \( u = \delta b \) then \( 0 = u(1, 1) = \delta b(1, 1) = bd(1, 1) = b(1) \), which proves \( b \in B^2_0 \), hence the map in the claim is injective.