Math 210A. Homework 9

1. Do Exercise 23 in Chapter XX, and as an application prove that $\mathbf{Z}/n\mathbf{Z}$ is injective as a module over itself and that $\mathbf{Q}/\mathbf{Z}$ is injective as a $\mathbf{Z}$-module. Also prove that $\mathbf{Z}$ is not injective as a module over itself.

2. Let A be a commutative ring, S a multiplicative set in A. An A-module M is A-*flat* if $M \otimes_A (\cdot)$ is exact.

(i) For any $S^{-1}A$-module M and A-module N, construct a natural A-linear isomorphism $M \otimes_A N \simeq M \otimes_{S^{-1}A} S^{-1}N$ (functorial in both M and N).

(ii) Prove that an A-module M is A-flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all prime ideals $\mathfrak{p}$ of A, and also if and only if all $M_{\mathfrak{p}}$ are A-flat. Show also that it suffices to consider just maximal ideals.

3. This exercise partially addresses the "symmetric" nature of Tor. (A systematic development via "bifunctors" is in the text, but is not needed at our present level.) Let R be a commutative ring.

(i) For any map of R-modules $f : M' \to M$, prove via universality considerations that there is a unique map of δ-functors $\mathrm{Tor}^{\bullet}_R(M', \cdot) \to \mathrm{Tor}^{\bullet}_R(M, \cdot)$ such that in degree 0 it is the natural map $f \otimes \mathrm{id}$. Deduce via uniqueness that for fixed N this makes $\mathrm{Tor}^i_R(M, N)$ a covariant additive functor in M for each i. Show it can also be computed concretely as follows: for a projective resolution $P^{\bullet} \to N$ show that $f \otimes \mathrm{id} : M' \otimes_R P^{\bullet} \to M \otimes_R P^{\bullet}$ induces on homologies exactly the map $\mathrm{Tor}^{\bullet}_R(M', N) \to \mathrm{Tor}^{\bullet}_R(M, N)$ constructed by universality considerations. (Hint: show that this more "concrete" approach defines a map of δ-functors which is as expected in degree 0.)

(ii) By (i), $\mathrm{Tor}^i_R(M, N)$ is an additive covariant functor in both M and N separately for each i. Using the description with projective resolutions in (i), construct a δ-functor structure in the first variable (when the second, N, is held fixed). Prove from the construction that $\mathrm{Tor}^i_R(M, N) = 0$ for $i > 0$ when M is a free module, and so deduce that the δ-functor structure in the first variable is *erasable*.

(iii) Using Grothendieck's theorem on erasable δ-functors, deduce that for each R-module M there is a *unique* isomorphism of δ-functors $\mathrm{Tor}^{\bullet}_R(M, \cdot) \simeq \mathrm{Tor}^{\bullet}_R(\cdot, M)$ which is the flip isomorphism $M \otimes N \simeq N \otimes M$ in degree 0. Conclude that $\mathrm{Tor}^i_R(M, N) = 0$ for all $i > 0$ when N is *flat*!

(iv) Using the conclusion of (iii), prove that if $C^{\bullet}$ is an exact sequence *flat* R-modules with $C^n = 0$ for $n \gg 0$ then $M \otimes_R C^{\bullet}$ is exact for any R-module M. (Hint: by descending induction on degree, show that $C^{\bullet}$ can be spliced into many short exact sequences of flats, so reducing to the case when $C^{\bullet}$ is a short exact sequence. Then use (iii).)

4. Let $\mathscr{C}$ be an abelian category with enough injectives. Consider the δ-functor $\mathrm{Ext}^{\bullet}_{\mathscr{C}}(M, \cdot)$ for $M \in \mathscr{C}$.

(i) Adapt the techniques in Exercise 3 to give $\mathrm{Ext}^i_{\mathscr{C}}(\cdot, N)$ an additive contravariant functor structure for each i with N fixed, and then give the entire collection a δ-functor structure for each N.

(ii) Assuming that $\mathscr{C}$ has enough projectives too, establish erasability with projectives in the first variable for the δ-functor in (i), and deduce that the method in (i) recovers the derived functor of $\mathrm{Hom}_{\mathscr{C}}(\cdot, N)$. It is really useful for some $\mathscr{C}$ that (i) works even without enough projectives.

5. Let G be a group.

(i) Prove that there is a unique δ-functorial isomorphism $\mathrm{H}^{\bullet}(G, M) \simeq \mathrm{Ext}^{\bullet}_{\mathbf{Z}[G]}(\mathbf{Z}, M)$ that in degree 0 is the isomorphism $M^G \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M)$. In particular, by Exercise 4(ii), we can compute G-cohomology via a *fixed* $\mathbf{Z}[G]$-projective resolution of $\mathbf{Z}$, independent of M! (There is a specific such resolution of $\mathbf{Z}$ that pervades the theory, the "standard cochain resolution", which is fantastically useful for doing concrete calculations. See Exercises 1–6 in Ch. XX or the handout on the "bar resolution" for how this goes.)

(ii) Assume G is finite. Explain why $\mathbf{Z}$ admits a projective resolution over $\mathbf{Z}[G]$ consisting of finite free left $\mathbf{Z}[G]$-modules. Conclude via (i) that $\mathrm{H}^i(G, M)$ is finitely generated as a $\mathbf{Z}$-module when G is finite and M is finitely generated as a $\mathbf{Z}$-module; this is certainly not obvious from the "injective resolution" viewpoint.

(iii) Assume G is finite of size $n > 0$. Show that for any G-module M, the composite of the inclusion $M^G \to M$ and the map $M \to M^G$ defined by $m \mapsto \sum_{g \in G} g.m$ is multiplication by n. By universality of derived functors, deduce that multiplication by n on $\mathrm{H}^i(G, M)$ factors through the ith derived functor of the exact functor $M \rightsquigarrow M$ from G-modules to abelian groups. Conclude that $\mathrm{H}^i(G, M)$ is killed by n for all $i > 0$, and that it is even *finite* when $i > 0$ and M is $\mathbf{Z}$-finite.