Math 210A. Homework 8

1. Read the handout on tensor algebras, highlights of which were discussed in class.

2. Let $R = k[X, Y]$ for a field k, and let $I = (X, Y)$. View k as an R-module via $k = R/I$.

(i) Prove that there is a unique R-linear map $I \otimes_R I \to k$ satisfying $f \otimes g \mapsto (\partial_X f)(0,0) \cdot (\partial_Y g)(0,0)$, and deduce that $X \otimes Y \neq Y \otimes X$ in $I \otimes_R I$.

(ii) As an application, show that the natural map $I \otimes_R I \to R \otimes_R R \simeq R$ is *not* injective. So the inclusion $I \to R$ considered twice is an example of a pair of injective linear maps T and T' between torsion-free modules such that $T \otimes T'$ is not injective.

3. Let R be a commutative ring, and $M_1, \dots, M_n$ be finite free R-modules. Construct and uniquely characterize (via elementary tensors) a linear isomorphism

$$M_1^\vee \otimes \cdots \otimes M_n^\vee \simeq (M_1 \otimes \cdots \otimes M_n)^\vee$$

recovering the one from class for $n = 2$, and discuss its naturality in $M_1, \dots, M_n$ (draw and justify a commutative diagram).

4. Let M_1, M_2, M_3 be finite free modules over a commutative ring R.

(i) Formulate and prove a naturality property for the isomorphism $M_2 \otimes M_1^\vee \simeq \mathrm{Hom}(M_1, M_2)$.

(ii) Prove that the composite isomorphism

$$\mathrm{Hom}(M_1 \otimes M_2, M_3) \simeq \mathrm{Hom}(M_1, \mathrm{Hom}(M_2, M_3)) \simeq \mathrm{Hom}(M_1, M_3 \otimes M_2^\vee)$$

(first step being $T \mapsto (m_1 \mapsto (m_2 \mapsto T(m_1 \otimes m_2))))$ coincides with the composite isomorphism

$$\mathrm{Hom}(M_1 \otimes M_2, M_3) \simeq M_3 \otimes (M_1 \otimes M_2)^\vee \simeq M_3 \otimes (M_1^\vee \otimes M_2^\vee) \simeq (M_3 \otimes M_2^\vee) \otimes M_1^\vee \simeq \mathrm{Hom}(M_1, M_3 \otimes M_2^\vee).$$

Formulate and prove a naturality statement. (Hint: to compare the isomorphisms, begin with an elementary tensor in the middle and chase it out to both ends.)

(iii) Using $M_1^\vee \otimes M_1 \to R$ via evaluation, prove that the composite map

$$\mathrm{Hom}(M_1, M_2) \otimes M_1 \simeq (M_2 \otimes M_1^\vee) \otimes M_1 \simeq M_2 \otimes (M_1^\vee \otimes M_1) \to M_2 \otimes R \simeq M_2$$

is characterized by $T \otimes m_1 \mapsto T(m_1)$.

5. Let M and M' be finite free modules over a commutative ring R, and let $\phi : M^\vee \otimes M' \simeq M'^{\vee\vee} \otimes M^\vee$ be

$$M^\vee \otimes M' \simeq M' \otimes M^\vee \simeq M'^{\vee\vee} \otimes M^\vee,$$

the first step being the "flip" isomorphism. By chasing suitable elementary tensors in $M^\vee \otimes M'$ (and *not* using bases), prove the commutativity of

$$\begin{array}{ccc} M^\vee \otimes M' & \xrightarrow{\ \simeq\ } & \mathrm{Hom}(M, M') \\ {\scriptstyle \phi}\big\downarrow & & \simeq \big\downarrow {\scriptstyle T \mapsto T^\vee} \\ M'^{\vee\vee} \otimes M^\vee & \xrightarrow[\ \simeq\ ]{} & \mathrm{Hom}(M'^\vee, M^\vee) \end{array}$$

6. Consider finite-dimensional vector spaces over a field F. Let W be a subspace of V with dimension $n > 0$. Prove that the line $\wedge^n(W)$ inside of $\wedge^n(V)$ uniquely determines W as a subspace of V. (Hint: if W' is another n-dimensional subspace, compute using a basis of V built from extending a basis of $W \cap W'$ separately to bases of W and W'.)