Math 210A. Homework 7

1. Read the handout on tensorial maps, and do the following exercises involving a commutative ring R.

(i) Prove that there is a unique linear isomorphism $R \otimes_R R \simeq R$ satisfying $a \otimes b \mapsto ab$.

(ii) Let M, M', N, M' be R-modules. Consider linear maps $T : M \to N$ and $T' : M' \to N'$. These define a linear map $T \otimes T' : M \otimes_R M' \to N \otimes_R N'$ uniquely characterized by $m \otimes m' \mapsto T(m) \otimes T'(m')$. Show that there is a unique R-linear map $\mathrm{Hom}_R(M, N) \otimes_R \mathrm{Hom}_R(M', N') \to \mathrm{Hom}_R(M \otimes_R M', N \otimes_R N')$ carrying the *elementary tensor* $T \otimes T'$ to the *linear map* denoted above by the same notation! Prove that if M, N, M', and N' are finite free R-modules then it is an isomorphism. Interpret the case $N = N' = R$ using (i).

(iii) For any four R-modules $M_1, \dots, M_4$ (not necessarily finite free), prove the existence and uniqueness of an R-linear isomorphism $(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \simeq M_1 \otimes ((M_2 \otimes M_3) \otimes M_4)$ satisfying $(m_1 \otimes m_2) \otimes (m_3 \otimes m_4) \mapsto m_1 \otimes ((m_2 \otimes m_3) \otimes m_4)$ for all $m_i \in M_i$.

(iv) For any R-module N, an element of $N \otimes N$ is *symmetric* if it is invariant under the "flip" automorphism characterized by $n \otimes n' \mapsto n' \otimes n$. Let M be a finite free R-module, and consider the composite linear isomorphism $M^\vee \otimes M^\vee \simeq (M \otimes M)^\vee \simeq \mathrm{Bil}(M \times M, R)$ (the final term given the natural R-module structure via pointwise linear combinations of bilinear forms). By working with elementary tensors, check that the "flip" automorphism on the left is carried over to the automorphism of the right defined by swapping variables (i.e., B is carried to $(m, m') \mapsto B(m', m)$). Deduce from this (without any explicit mention of bases) that symmetric bilinear forms on the right correspond to symmetric tensors on the left and vice-versa.

2. For ideals I, J in a commutative ring R, show $(R/I) \otimes_R (R/J) \simeq R/(I + J)$ via $a \otimes b \mapsto ab \bmod I + J$. Deduce for $n, m \geq 1$ that $(\mathbf{Z}/n\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/m\mathbf{Z}) = 0$ precisely when $\gcd(m, n) = 1$. Also prove directly that there is no nonzero $\mathbf{Z}$-bilinear map $(\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/m\mathbf{Z}) \to M$ to a $\mathbf{Z}$-module M when $\gcd(m, n) = 1$.

3. Let A be a commutative ring, and M an A-module.

(i) For any multiplicative set S in A, prove the existence and uniqueness of an A-linear isomorphism $S^{-1}A \otimes_A M \simeq S^{-1}M$ satisfying $(a/s) \otimes m \mapsto am/s$.

(ii) Let B be an A-algebra, and M an A-module. Prove that the A-module $B \otimes_A M$ admits a *unique* B-module structure respecting its A-module structure and satisfying $b' \cdot (b \otimes m) = b'b \otimes m$ for all $b, b' \in B$ and $m \in M$. This is called the *scalar extension* of M to B. In case $B = A/I$ show that this is the usual A/I-module structure on M/IM via the A-linear isomorphism $(A/I) \otimes_A M \simeq M/IM$, and show that it makes the isomorphism in (i) become $S^{-1}A$-linear.

(iii) Let $T : M' \to M$ be an A-linear map, and B an A-algebra. Prove that there is a unique B-linear map $T_B : B \otimes_A M' \to B \otimes_A M$ satisfying $b \otimes m' \mapsto b \otimes T(m)$. This is called the *scalar extension* of T over B. For A-linear $T' : M'' \to M'$ prove $(T \circ T')_B = T_B \circ T'_B$, and explain how scalar extension is related to the ring map $\mathrm{Mat}_{n \times n'}(A) \to \mathrm{Mat}_{n \times n'}(B)$.

(iv) Let $A \xrightarrow{g} B \xrightarrow{f} C$ be ring maps, and $T : M' \to M$ an A-linear map. Prove the existence and uniqueness of a C-linear isomorphism $C \otimes_B (B \otimes_A M) \simeq C \otimes_A M$ satisfying $c \otimes (b \otimes m) \mapsto cf(b) \otimes m$, and show it is functorial in M in the sense that it carries $(T_B)_C$ over to T_C.

4. Let X and X' be objects in a category C, and let $h_X = \mathrm{Hom}_C(\cdot, X)$ and $h_{X'} = \mathrm{Hom}_C(\cdot, X')$ be the associated contravariant functors from C to the category of sets.

(i) For any morphism $f : X' \to X$, define a natural transformation $h_f : h_{X'} \to h_X$ via composition with f. Check that this really is a natural transformation, that h_{id_X} is the identity transformation of h_X, and that $h_{f \circ g} = h_g \circ h_f$ for any $g : X'' \to X'$.

(ii) Prove *Yoneda's Lemma*: every natural transformation $h_{X'} \to h_X$ has the form h_f for a unique f. (Hint: chase identity morphisms of X and X'.) This simple-looking fact is extremely useful.

(iii) For a contravariant functor F from C to the category of sets, and $\xi \in F(X)$, define $h_X \to F$ via $\mathrm{Hom}_C(Y, X) \to F(Y)$ carrying $f : Y \to X$ to $F(f)(\xi) \in F(Y)$. (Here we used contravariance of F.) Prove that this is a natural transformation, and that every natural transformation $h_X \to F$ has this form for a unique $\xi \in F(X)$. In the special case that $h_X \to F$ is an isomorphism, we call the pair (X, ξ) – and not just X! – a *universal object* for F. (There is an evident analogue for covariant F.) How does this generalize (ii)?