Math 210A. Homework 6

1. Let C be a category, and $\{X_i\}_{i\in I}$ a set of objects in C.

(i) A *product* $\prod X_i$ in C (if it exists) is an object P equipped with maps $p_i : P \to X_i$ such that for any object X and collection of maps $f_i : X \to X_i$, there is a unique map $f : X \to P$ such that $p_i \circ f = f_i$ for all i. Express this condition as a final object in a category, and deduce that such a $(P, \{p_i\})$ is unique up to *unique* isomorphism (in what sense of "isomorphism"?) if it exists.

(ii) Passing to C^{opp}, we get a notion of *coproduct* of the X_i's. Express this concretely in terms of C (in terms of an initial object for receiving maps from all X_i's). Prove that in the category of sets coproducts exist and are disjoint unions (with which maps from the X_i's?), and that in the category of R-modules coproducts exist and are direct sums (with which maps from the X_i's?), so "underlying set" does not commute with formation of coproducts here (in contrast with products)!

2. Let D be a diagram of objects X_i in a category C. (There could be many maps between the same pair of objects.) Consider the tuples $(X, \{f_i\})$ consisting of maps $f_i : X \to X_i$ from a single X such that the f_i's are compatible with *all* maps in D: for any map $h : X_i \to X_j$ occurring in D, $h \circ f_i = f_j$.

(i) Define a reasonable notion of morphism between such tuples so that you get a category, and formulate in concrete terms what a final object means in this category. Such an object (along with its maps to the X_i's!) is called an *inverse limit* of D if it exists, and is denoted $\varprojlim D$. Important cases are (a) D consists of a sequence of objects $\{X_n\}_{n\geq 0}$ equipped with maps $f_n : X_{n+1} \to X_n$ for all n, for which $\varprojlim D$ is denoted $\varprojlim X_n$ with the f_n's understood, (b) no maps in D, in which case $\varprojlim D$ is just $\prod X_i$ by another name!

(ii) Using submodules of products, show that inverse limits always exist in the category of modules over a ring. Do similarly in the category of rings. And the category of sets.

(iii) Let A be a commutative ring, and I an ideal. Consider the diagram using $A/I^{n+1} \to A/I^n$ for all n. Construct a ring map $A \to \varprojlim A/I^n$, and show A is adic with respect to I if and only if it is an isomorphism.

3. (i) By transferring the notion of inverse limit from C^{opp} as in Exercise 2, explicitly define the concept of a *colimit* of a diagram D in C without mentioning C^{opp}; , it is denoted $\varinjlim D$ (if it exists).

(ii) Using quotients of direct sums, construct colimits in the category of modules. Also do the construction in the category of sets, using the quotient of a disjoint union by a suitable equivalence relation.

(iii) Prove that a module is a colimit of the diagram (with inclusion maps) of finitely generated submodules.

4. Read the short §3 in Chapter III, and do the following.

(i) Using AB1, prove that zero in $\text{Hom}(X, Y)$ is the composite $X \to 0 \to Y$ in additive categories. For a finite coproduct $\oplus X_i$ and finite product $\prod X_i$, prove there is a unique map $f : \oplus X_i \to \prod X_i$ such that $X_{i_0} \to \oplus X_i \xrightarrow{f} \prod X_i \to X_{i_0}$ is the identity for all i_0. Axiom AB2 should require f to be an isomorphism!

(ii) In additive categories, show kernels and cokernels have the categorical characterizations as for modules. For $f : M \to M'$, prove $\ker f = 0$ if and only if $\text{Hom}(X, M) \to \text{Hom}(X, M')$ is injective for all X, and $\text{coker}\, f = 0$ if and only if $\text{Hom}(M', X) \to \text{Hom}(M, X)$ is injective for all X (the respective definitions of *monomorphism* and *epimorphism*). If C is abelian, prove C^{opp} is, swapping kernels and cokernels.

(iii) In the category of finite free $\mathbf{Z}$-modules, prove kernels and cokernels exist with $f : M \to M'$ having cokernel $(M'/f(M))/(M'/f(M))_{\text{tor}}$. Exhibit f with $\ker f = 0$ and $\text{coker}\, f = 0$ yet f not an isomorphism, so C is not abelian, Is the category of complexes of modules over an associative ring abelian?

(iv) Let $f' : M \to M'$ and $f'' : M \to M''$ be morphisms in an abelian category. Let $\delta : M \to M' \oplus X$ be the "anti-diagonal" given by $(f', -f'')$. Prove that $P := \text{coker}\, \delta$ *equipped with* its natural maps from M' and M'' is a colimit of the diagram $M' \xleftarrow{f'} M \xrightarrow{f''} M''$ in the sense of Exercise 3(i); it is called a *pushout* (of f' along f'', or vice-versa). Using AB4, show that if f' is a monomorphism then so is $M'' \to P$.

(v) For $f : M \to M'$ in an additive category with AB3, the *coimage* is $\text{coim}(f) = \text{coker}(\ker f \to M)$ and the *image* is $\text{im}(f) = \ker(M' \to \text{coker}\, f)$. Prove $\text{coim}(f) \to M'$ uniquely factors through $\text{im}(f) \to M'$, and $\text{coim}(f) \to \text{im}(f)$ is a monomorphism when AB4 holds (apply end of (iv) to pushout of $\text{im}(f) \to M'$ along any $\text{im}(f) \to X$). Use the end of (ii) to infer $\text{coim}(f) \to \text{im}(f)$ is an isomorphism in *abelian categories*.

5. Let $\{v_i\}$ be a basis of a finite-dimensional vector space V over a field k. Prove that $x = \sum c_{ij} v_i \otimes v_j \in V \otimes V$ is an elementary tensor (i.e., $x = v \otimes w$ for some $v, w \in V$) if and only if $c_{ij} c_{i'j'} = c_{ij'} c_{i'j}$ for all i, j, i', j'.