Math 210A. Homework 5

1. Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a diagram of A-modules (A a commutative ring). Prove that it is exact if and only if the localized diagram of $A_{\mathfrak{p}}$-modules is exact for any prime ideal $\mathfrak{p}$ of A. Show that it even suffices to check exactness after localization at all maximal ideals of A. This is very important in algebraic geometry.

2. Consider the setup of the snake lemma: a commutative diagram of A-modules

$$\begin{array}{ccccccc} & & M' & \xrightarrow{g_1} & M & \xrightarrow{h_1} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow[g_2]{} & N & \xrightarrow[h_2]{} & N'' & & \end{array}$$

In class we constructed an associated complex

$$\ker f' \to \ker f \to \ker f'' \xrightarrow{d} \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f''$$

and we claimed it is exact (and addressed it at $\ker f''$). Check the exactness at $\operatorname{coker} f'$ (it is trivial elsewhere).

3. Consider a short exact sequence $0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$ of complexes of A-modules.

(i) In class we associated to this a long exact sequence in homologies, conditional on the diagrams

$$M'_i/\mathrm{im}(f'_{i-1}) \to M_i/\mathrm{im}(f_{i-1}) \to M''_i/\mathrm{im}(f''_{i-1}) \to 0, \quad 0 \to \ker f'_{i+1} \to \ker f_{i+1} \to \ker f''_{i+1}$$

being exact sequences for all i. Verify such exactness.

(ii) The construction of the long exact sequence in homologies is "natural" in the sense that if we are given a commutative diagram of maps of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_\bullet & \longrightarrow & M_\bullet & \longrightarrow & M''_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N'_\bullet & \longrightarrow & N_\bullet & \longrightarrow & N''_\bullet & \longrightarrow & 0 \end{array}$$

with exact rows (i.e., exact for each fixed index), then the induced diagram of long exact sequences in homologies

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \mathrm{H}_{i-1}(M''_\bullet) & \xrightarrow{d} & \mathrm{H}_i(M'_\bullet) & \longrightarrow & \mathrm{H}_i(M_\bullet) & \longrightarrow & \mathrm{H}_i(M''_\bullet) & \xrightarrow{d} & \mathrm{H}_{i+1}(M'_\bullet) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}_{i-1}(N''_\bullet) & \xrightarrow{d} & \mathrm{H}_i(N'_\bullet) & \longrightarrow & \mathrm{H}_i(N_\bullet) & \longrightarrow & \mathrm{H}_i(N''_\bullet) & \xrightarrow{d} & \mathrm{H}_{i+1}(N'_\bullet) & \longrightarrow & \cdots \end{array}$$

commutes. Verify such commutativity at the squares involving the connecting maps d (the others are trivial).

4. (i) Formulate and prove a precise result asserting that the construction of the long exact sequence (especially the connecting map) in the snake lemma is compatible with localization at any multiplicative set.

(ii) Do the same for the formation of the diagram of homologies in Exercise 3(ii).

5. Consider a commutative diagram of A-modules with exact rows:

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

The following is the "five lemma". (Do not memorize it! Rederive it every time you need it.)

(i) Prove that f_3 is injective if f_1 is surjective and f_2 and f_4 are injective.

(ii) Prove that f_3 is surjective if f_5 is injective and f_2 and f_4 are surjective (so as an important special case, if f_1, f_2, f_4, and f_5 are isomorphisms then so is f_3).