Math 210A. Homework 4

1. Read the handout on Eisenstein's criterion and Gauss' Lemma. Then do the following exercise at the end of that handout: factor $-6x^3 + 6x^2y^2 + 6x^3y - 3xy + 3y^3 + 3xy^2 \in \mathbf{Z}[x, y]$ into irreducibles.

2. Let A be a commutative ring, $a \in A$ any element. Recall that $A[x]/(ax - 1) \to A_a$ defined by $x \mapsto 1/a$ is an isomorphism. Deduce that $1 - ax \in A[x]^\times$ if and only if a is nilpotent. In such cases, give a formula for $(1 - ax)^{-1}$ in $A[x]$. (Optional: can you prove the "only if" direction without using the ring A_a?)

3. Let A be a commutative ring, and S and T multiplicative sets in A with $S \subseteq T$.

(i) Prove that there is a unique A-algebra map $S^{-1}A \to T^{-1}A$. In case $T = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of A disjoint from S, with $P = S^{-1}\mathfrak{p}$ the prime ideal of $S^{-1}A$ corresponding to $\mathfrak{p}$, show that the map $S^{-1}A \to A_\mathfrak{p}$ uniquely factors through an isomorphism $(S^{-1}A)_P \simeq A_\mathfrak{p}$.

(ii) Show that under the map of rings $A \to A_\mathfrak{p}$, the preimage of the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ is $\mathfrak{p}$, so there is a natural injective map $A/\mathfrak{p} \hookrightarrow A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ into the residue field at $\mathfrak{p}A_\mathfrak{p}$. Deduce that there is a natural map of fields $\mathrm{Frac}(A/\mathfrak{p}) \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, and prove it is an isomorphism inverse to the isomorphism constructed in class.

4. Let A be a commutative ring, S a multiplicative set of A, and M and N two A-modules.

(i) Construct a natural $S^{-1}A$-linear map $S^{-1}\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$, and prove it is an isomorphism if M is a free A-module of finite rank. Deduce the same if M admits a "finite presentation" $M \simeq \mathrm{coker}(A^n \to A^{n'})$ for some $n, n' \geq 1$; i.e., M has a finite set of generators whose module of relations is also finitely generated. (Hint: if $M' \twoheadrightarrow M$ is surjective with kernel M'' then show that $\mathrm{Hom}_A(M, N)$ is the kernel of the restriction map $\mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M'', N)$.) In Math 210B you will see *many* examples of A for which all finitely generated A-modules admit a finite presentation.

(ii) Prove that the natural maps $\oplus M_i \to \oplus S^{-1}M_i$ and $\prod M_i \to \prod S^{-1}M_i$ uniquely factor through $S^{-1}A$-linear maps $S^{-1}(\oplus M_i) \to \oplus S^{-1}M_i$ and $S^{-1}(\prod M_i) \to \prod S^{-1}M_i$ respectively, with the first an isomorphism. Give an example in which the second is not surjective, and an example in which it is not injective.

(iii) Let $M_1, \ldots, M_n$ be submodules of M. Prove that $\cap S^{-1}M_i = S^{-1}(\cap M_i)$ inside of $S^{-1}M$. Give a counterexample with infinite intersections.

5. Give an example of a commutative ring A and an A-module M such that $M_\mathfrak{p}$ is finitely generated over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A, but M is not finitely generated over A. (Hint: try $A = \mathbf{Z}$.) Prove the following salvage: if $a_1, \ldots, a_n \in A$ generate the unit ideal and M_{a_i} is finitely generated over A_{a_i} for all i then show that M is finitely generated over A. (Hint: chase numerators.)

6. Let A be a commutative ring, and let $\mathrm{Spec}\, A$ denote the set of prime ideals of A. (For example, $\mathrm{Spec}\, \mathbf{Z}$ consists of (0) and the ideals (p) for positive primes p.)

(i) For any ideal I of A, let $V(I) \subseteq \mathrm{Spec}\, A$ denote the set of prime ideals containing I. (Note that $V(I^n) = V(I)$ for any $n \geq 1$, so $V(I)$ does not determine I.) Prove that $V((1)) = \emptyset$, $V((0)) = \mathrm{Spec}\, A$, $V(I_1) \cup \cdots \cup V(I_n) = V(\prod I_j)$, and $\cap V(I_i) = V(\sum I_i)$ for any collection of ideals $\{I_i\}$ in A. In other words, there is a unique topology on $\mathrm{Spec}\, A$ relative to which the subsets $V(I)$ are the closed sets; this is the *Zariski topology*.

(ii) The open sets in the Zariski topology are "big". Prove that a base of opens is given by the subsets $U_a = \mathrm{Spec}(A) - V((a))$ for $a \in A$ ("complement of a hypersurface"), and that a collection of opens $\{U(a_i)\}$ covers $\mathrm{Spec}\, A$ if and only if the a_i's generate 1. Deduce that $\mathrm{Spec}\, A$ is quasi-compact (i.e., every open cover has a finite subcover).

(iii) Let $f : A \to B$ be a map of rings. Prove that the map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ defined by $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$ is continuous, and that the preimage of $\mathfrak{p} \in \mathrm{Spec}\, A$ is homeomorphic to $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ (where $B_\mathfrak{p}$ is the localization of B at the multiplicative set $f(A - \mathfrak{p})$).

(iv) Applying (iii) to $A \to S^{-1}A$, show that $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}\, A$ is a homeomorphism onto the subset of primes disjoint from S. In particular, $\mathrm{Spec}(A_a)$ maps homeomorphically onto the open subset U_a.