Math 210A. Homework 3

1. (i) Prove that if a nonzero ideal I in a domain R is free as an R-module then I is principal. As an application, for $R = \mathbf{Z}[\sqrt{-5}]$ prove that neither of the ideals $P = (3, 1+\sqrt{-5})$ and $Q = (3, 1-\sqrt{-5})$ is free.

(ii) Prove that $P \cap Q = 3R$, and that the addition map $P \oplus Q \to R$ defined by $(a, b) \mapsto a + b$ is surjective.

(iii) Deduce that $P \oplus Q \simeq R^2$ as R-modules, so a direct summand of a free module need not be free!

2. (i) Read §2 in Appendix 2 on Zorn's Lemma.

(ii) As an application, prove that if A is a nonzero commutative ring then there exist maximal ideals (and in particular, prime ideals) in A. (Where does your argument use $A \neq 0$?) Applying this to $A/\mathfrak{a}$ for a proper ideal $\mathfrak{a}$ of A, prove that $\mathfrak{a}$ is contained in a maximal ideal of A.

(iii) Using the operation $M \rightsquigarrow M/\mathfrak{m}M$ for A-modules M and maximal ideals $\mathfrak{m}$ of A, prove that if $A \neq 0$ and there is an A-linear surjection (resp. isomorphism) $A^n \twoheadrightarrow A^m$ then $n \geq m$ (resp. $n = m$). Deduce that if a module over a nonzero commutative ring admits a finite basis then all bases have the same finite size (called the *rank* of the module).

3. Let V be a finite-dimensional nonzero vector space over a field F. A linear self-map $T : V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace. (That is, if $T(W) \subseteq W$ then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g., consider T to be a scalar multiplication with $\dim V > 1$).

(i) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of $v \in V$ killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if $\pi | m_T$, and that $V = \oplus_{\pi | m_T} V(\pi)$. (In case F is algebraically closed, these are the *generalized eigenspaces* of T on V.)

(ii) Use rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over F. (Hint: apply (i) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that a Jordan block of rank > 1 is never semisimple, that m_T is the "squarefree part" of χ_T when T is semisimple, and that if $W \subseteq V$ is a T-stable nonzero proper subspace then the induced endomorphisms $T_W : W \to W$ and $\overline{T} : V/W \to V/W$ are semisimple when T is semisimple.

(iii) Let $T' : V' \to V'$ be another linear self-map with V' nonzero and finite-dimensional over F. Prove that T and T' are semisimple if and only if the self-map $T \oplus T'$ of $V \oplus V'$ is semisimple.

(iii) Choose $T \in \mathrm{Mat}_n(F)$, and let F'/F be an extension splitting m_T. Prove that T is semisimple as an F'-linear endomorphism of ${F'}^n$ if and only if T is diagonalizable over F', and also if and only if $m_T \in F[t]$ is separable; we then say T is *absolutely semisimple* over F. Deduce that semisimplicity is equivalent to absolutely semisimplicity over F if F is perfect, and give a counterexample over *every* imperfect field.

4. Let V be a vector space over a field F with $n = \dim V > 0$ finite, and let $T : V \to V$ be linear.

(i) Using rational canonical form and Cayley-Hamilton, prove the following are equivalent: $T^N = 0$ for some $N \geq 1$, $T^n = 0$, with respect to some ordered basis of V the matrix for T is upper triangular with 0's on the diagonal, $\chi_T = t^n$. We call such T *nilpotent*.

(ii) We say that T is *unipotent* if $T - 1$ is nilpotent. Formulate characterizations of unipotence analogous to the conditions in (i), and prove that a unipotent T is invertible.

(iii) Assume F is algebraically closed. Using Jordan canonical form and generalized eigenspaces, prove that there is a unique expression $T = T_{\mathrm{ss}} + T_{\mathrm{n}}$ where T_{ss} and T_{n} are a pair of *commuting* endomorphisms of V with T_{ss} semisimple and T_{n} nilpotent. (This is the *additive Jordan decomposition* of T.) Show by example with $\dim V = 2$ that uniqueness fails if we drop the "commuting" requirement, and show in general that $\chi_T = \chi_{T_{\mathrm{ss}}}$ (so T is invertible if and only if T_{ss} is invertible).

(iv) Assume F is algebraically closed and T is invertible. Using the existence and uniqueness of additive Jordan decomposition, prove that there is a unique expression $T = T'_{\mathrm{ss}} T'_{\mathrm{u}}$ where T'_{ss} and T'_{u} are a pair of *commuting* endomorphisms of V with T'_{ss} semisimple and T'_{u} unipotent (so T'_{ss} is necessarily invertible too). This is the *multiplicative Jordan decomposition* of T.

(v) Use Galois theory with matrices to prove (iii) and (iv) for any perfect F (using the result over an algebraic closure, or rather over a suitable finite Galois extension), and give counterexamples for any imperfect F. This leads to the important *Jordan decomposition* in Lie algebras and linear algebraic groups.