Math 210A. Homework 2

1. Let R be a commutative ring.

(i) Rigorously define $R[\![X_1,\dots,X_n]\!]$ (*formal power series* over R) akin to the case of polynomial rings, and describe the kernel of the surjective map $R[\![X_1,\dots,X_n]\!] \twoheadrightarrow (R/I)[\![X_1,\dots,X_n]\!]$ for ideals I in R.

(ii) A commutative ring A is *adic* with respect to an ideal J if any compatible sequence of elements $a_i \in A/J^i$ (i.e., $A/J^{i'} \twoheadrightarrow A/J^i$ carries $a_{i'}$ to a_i when $i' \geq i$) has the form $a_i = a \bmod J^i$ for a unique $a \in A$. (Taking $a_i = 0$ for all i, $\cap J^i = 0$.) Prove $R[\![X_1,\dots,X_n]\!]$ is adic with respect to $J = (X_1,\dots,X_n)$.

(iii) Prove that if A is adic with respect to J then $a \in A$ lies in $A^\times$ if and only if $a \bmod J \in (A/J)^\times$. (Hint: reduce to the case $a \bmod J = 1$ and consider the formal series expansion for $(1+t)^{-1}$.) Deduce that $f \in R[\![X_1,\dots,X_n]\!]$ is a unit if and only if $f(0) \in R^\times$.

(iv) If A is an R-algebra that is adic with respect to an ideal J, and $x_1,\dots,x_n \in J$, prove there exists a unique map of R-algebras $R[\![X_1,\dots,X_n]\!] \to A$ satisfying $X_i \mapsto x_i$ for all i. Deduce that a self-map ϕ of $R[\![X_1,\dots,X_n]\!]$ as an R-algebra carrying $(X_1,\dots,X_n)$ into itself has the form $f(X_1,\dots,X_n) \mapsto f(\phi_1,\dots,\phi_n)$ for $\phi_1,\dots,\phi_n$ satisfying $\phi_i(0) = 0$, and that ϕ is an isomorphism if and only if $\det((\partial\phi_i/\partial X_j)(0)) \in R^\times$.

2. An ideal I in a commutative ring R is *maximal* if $I \neq R$ and there are no ideals strictly between I and R, and is *prime* if $I \neq R$ and $ab \in I$ implies $a \in I$ or $b \in I$ (for $a, b \in R$).

(i) Prove I is maximal if and only if R/I is a field, and I is prime if and only if R/I is a domain. (In particular, every maximal ideal is prime.)

(ii) Prove that in a PID, nonzero prime ideals are maximal and are exactly the ideals generated by a prime element, and show that for any field k there are non-maximal nonzero prime ideals in $k[X,Y]$.

(iii) Let J be any ideal in R. Show that under the correspondence between ideals of R/J and ideals of R that contain J, maximals correspond to maximals (in both directions) and likewise for prime ideals.

(iv) Let $f : R' \to R$ be a ring homomorphism, and P a prime ideal if R. Prove that the preimage $f^{-1}(P)$ is a prime ideal of R', but show by example that if P is maximal then $f^{-1}(P)$ can fail to be maximal. (Hint: find a field containing a subring that is not a field.)

3. Read the statement and proof of the "Chinese Remainder Theorem" for commutative rings (and the immediate corollary following it in the course text. (Look up "Chinese Remainder Theorem" in the index to find it.) As an application, prove that if $\mathfrak{m}_1,\dots,\mathfrak{m}_n$ are pairwise distinct maximal ideals in a commutative ring R and $e_1,\dots,e_n \geq 1$ then the natural map $R/(\cap \mathfrak{m}_i^{e_i}) \to \prod (R/\mathfrak{m}_i^{e_i})$ is an isomorphism. (Taking $R = \mathbf{Z}$ and $\mathfrak{m}_i = p_i\mathbf{Z}$ for distinct positive primes p_i recovers the classical Chinese Remainder Theorem.)

4. Let R be a commutative ring. For R-modules M and N, define the R-module $\mathrm{Hom}_R(M,N)$ to be the set of R-linear maps $M \to N$ endowed with R-linear structure via pointwise operations (i.e., $(T_1+T_2)(m) = T_1(m) + T_2(m)$, $(r.T)(m) = r.T(m)$).

(i) Check that $\mathrm{Hom}_R(M,N)$ is an R-module, and give a counterexample with non-commutative R if we work with left R-modules throughout. Also explain how an R-linear map $L : M' \to M$ (resp. $L : N' \to N$) naturally induces an R-linear map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M',N)$ (resp. $\mathrm{Hom}_R(M,N') \to \mathrm{Hom}_R(M,N)$).

(ii) Taking $N = R$, the *dual* of M is $M^* = \mathrm{Hom}_R(M,R)$. In case R is a domain show that M^* is always torsion-free (even if M is not), and give an example with $R = \mathbf{Z}$ for which an injective map $M' \to M$ has associated dual map $M^* \to M'^*$ that is not surjective. Prove that the a surjective map $M' \to M$ induces an injection between duals.

5. Compute the units in $\mathbf{Z}[\sqrt{-3}]$ and $\mathbf{Z}[\zeta_3]$ (with $\zeta_3 = (-1+\sqrt{-3})/2$ a root of $x^2+x+1=0$), and prove that $3 \in \mathbf{Z}[\sqrt{-5}]$ is irreducible but not prime (hint: $3|(1+\sqrt{-5})(1-\sqrt{-5})$). In particular, $\mathbf{Z}[\sqrt{-5}]$ is not a PID. Give a justified example of a non-principal ideal in this ring.

6. Prove that if R is a domain and $a, b \in R$ satisfy $a|b$ and $b|a$ then $a \in R^\times \cdot b$. Give a counterexample with a non-domain R.