Math 210A. Homework 10

1. (i) Do Exercise 27 in Chapter XX. (Allow the coefficient ring to be associative.)

(ii) Here is an alternative, and more intrinsic, approach to the same thing: applying $\mathrm{Hom}(\cdot, N)$ to the given short exact sequence to get an exact sequence

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(E,N) \to \mathrm{Hom}(N,N) \xrightarrow{\delta} \mathrm{Ext}^1(M,N).$$

Prove $\mathrm{id}_N \in \mathrm{Hom}(N,N)$ is carried by δ to the *same* element of $\mathrm{Ext}^1(M,N)$. (A more interesting challenge, not assigned, is to describe the additive and inversion structure on the group $\mathrm{Ext}^1(M,N)$ entirely in the language of extension classes. This is useful. It is called *Baer sum.*)

2. Let G be an arbitrary group (not necessarily finite), and consider linear representations (not necessarily of finite dimension) over an arbitrary field k.

(i) If $V_1, \dots, V_n$ are representation spaces, prove that $V_1 \otimes \cdots \otimes V_n$ has a unique k-linear G-action such that $g.(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n)$ for all $g \in G$ and $v_i \in V_i$. Using functoriality considerations, explain why $V_1 \otimes V_2 \simeq V_2 \otimes V_1$ and the "associativity" of tensor products with $n = 3$ are G-equivariant.

(ii) Verify that if (W, ρ) is a representation space, then so is its linear dual W^* via $g.\ell = \ell \circ \rho(g^{-1})$. Prove that this is the unique action that makes the evaluation mapping $W \otimes W^* \to k$ be G-equivariant when using the trivial G-action on k.

(iii) Let $B : W \times W' \to k$ be a bilinear pairing between representation spaces. Prove that B is G-invariant in the sense that $B(gw, gw') = B(w, w')$ for all w, w', g if and only if the induced linear map $W \otimes W' \to k$ is G-equivariant (using the trivial action on k), and also if and only if the induced linear map $W' \to W^*$ (via $w' \mapsto B(\cdot, w')$) is G-equivariant. (For example, by (ii), $W \to W^{**}$ is G-equivariant.)

(iv) Construct natural G-actions on $\mathrm{Sym}^n(V)$ and $\wedge^n(V)$ for all $n \geq 1$, and if $\dim V$ is finite prove that the natural duality isomorphism $\wedge^n(V^*) \simeq (\wedge^n V)^*$ and its tensor product analogue are G-equivariant.

3. Let G be an arbitrary group and (V, ρ) and (V', ρ') be representation spaces (possibly of infinite dimension) over an arbitrary field k. Define a G-action on the k-vector space $\mathrm{Hom}_k(V, V')$ by $g.T = \rho'(g) \circ T \circ \rho(g^{-1})$.

(i) Prove this really is a G-linear representation structure (in particular, a *left* G-action) on the k-vector space $\mathrm{Hom}(V, V')$, with $\mathrm{Hom}(V, V')^G = \mathrm{Hom}_{k[G]}(V, V')$.

(ii) If $\dim V$ is finite, prove the linear isomorphism $V' \otimes V^* \simeq \mathrm{Hom}_k(V, V')$ is G-equivariant.

4. Let V be a finite-dimensional representation over any field k for an arbitrary group G.

(i) On the category of (left) $k[G]$-modules, identify $\mathrm{Hom}_{k[G]}(V, \cdot)$ and $(V^* \otimes_k (\cdot))^G$. (Use Exercise 3.)

(ii) Let $\mathrm{H}^\bullet_k(G, \cdot)$ denote the derived functor of $M \rightsquigarrow M^G$ on the category of left $k[G]$-modules. (It can be shown that this agrees with the restriction of $\mathrm{H}^\bullet(G, \cdot)$ computed on the category of left $\mathbf{Z}[G]$-modules, but we omit that here.) Use universal δ-functoriality to naturally identify $\mathrm{Ext}^i_{k[G]}(V, V')$ and $\mathrm{H}^i_k(G, V^* \otimes V')$ for all i. (This is especially useful for $i = 1$, due to Exercise 1, when generalized to allow k beyond fields.)

5. Let G_1 and G_2 be finite groups, and (V_i, ρ_i) a finite-dimensional representation of G_i over a field k with $\mathrm{char}(k) = 0$.

(i) Prove that $V = V_1 \otimes_k V_2$ is a representation of $G = G_1 \times G_2$ via $(g_1, g_2).(v_1 \otimes v_2) = (g_1.v_1) \otimes (g_2.v_2)$. (Don't confuse this with the setup in Exercise 2(i), where there was just one group!)

(ii) Let $\chi_i = \chi_{\rho_i}$ as a character of G_i and $\chi = \chi_V$ as a character of G. Prove that

$$\langle \chi, \chi \rangle_G = \langle \chi_1, \chi_1 \rangle_{G_1} \cdot \langle \chi_2, \chi_2 \rangle_{G_2}.$$

When k is algebraically closed, deduce that χ is irreducible for G if and only if each χ_i is irreducible for G_i. Give a counterexample to "if" with $k = \mathbf{R}$ (hint: consider cyclic G_i and $\dim V_i = 2$ with $V^G \neq 0$).

(iii) Assume k is algebraically closed. Prove that χ determines the χ_i's, and that every irreducible representation of $G_1 \times G_2$ over k arises in this way. (Hint: View a representation space for $G_1 \times G_2$ as one for each G_i separately. For the second part, if V_1 is an irreducible constituent of V for G_1 then show $\mathrm{Hom}_{k[G_1]}(V_1, V)$ is naturally a nonzero G_2-representation space. Let V_2 be an irreducible constituent and consider the evaluation map $V_1 \otimes_k V_2 \to V$; is it G-equivariant and nonzero?)