Math 210A. Isomorphisms with tensor products

1. Motivation

In this handout, we work out some interesting isomorphisms involving tensor products of modules. We will work with finite free modules over a ring. The ring will be denoted F and the finite free modules V and V', but this is just psychologically suggestive of the case of vector spaces. We will never need anything specific to the vector space case, but we will use module freeness in an essential way.

The three basic principles in working with maps involving tensor product modules are: (i) to *construct* maps involving tensor product modules we should never use bases (if they exist) or special spanning sets, and should instead let suitable bilinearity (or multilinearity) of formulas do all of the work, (ii) to prove *properties* of maps among tensor product modules we may have to use special spanning sets or bases (though not always; this is a matter of experience), and (iii) when checking identities among different ways of constructing linear maps, or proving that an abstract "tensorial" construction recovers some concrete construction (that also depends *linearly* on the input), it suffices to check by chasing elementary tensors.

Point (iii) merits further emphasis. In principle, when verifying an identity among *linear* maps between tensor product modules one can chase what happens on an arbitrary linear combination of elementary tensors; however, this often makes things a big mess, and so one should just chase elementary tensors (which suffice: they do span the tensor product modules, after all). Also, although one may note that in the spirit of cutting down one's work it is enough to even just chase what happens on elementary tensors from a single choice of spanning set (such as a single basis if one exists), by treating *all* elementary tensors in a uniform way one usually gets cleaner calculations without a profusion of subscripts.

2. Duality and tensors

For any finite free F-module V, the linear dual module $V^\vee = \mathrm{Hom}(V, F)$ is also finite free of the same rank, with an evident notion of dual basis akin to the case of vector spaces. Likewise, if V_1 and V_2 are finite free F-modules then so is $V_1 \otimes V_2$ (with basis given by elementary tensors in a choice of basis for each of V_1 and V_2). In particular, the dual module $(V_1 \otimes V_2)^\vee$ is also finite free. For such finite free F-modules V_1 and V_2, we saw in class how to define a unique linear map

$$V_1^\vee \otimes V_2^\vee \to (V_1 \otimes V_2)^\vee$$

satisfying

$$\ell_1 \otimes \ell_2 \mapsto (v_1 \otimes v_2 \mapsto \ell_1(v_1)\ell_2(v_2)).$$

Recall that the construction of this required two steps: first we had to check that for any pair $\ell_i \in V_i^\vee$ the proposed linear functional on $V_1 \otimes V_2$ (sending elementary tensors $v_1 \otimes v_2$ to the proposed value $\ell_1(v_1)\ell_2(v_2)$) made sense – this amounts to the fact that $\ell_1(v_1)\ell_2(v_2) \in F$ depends bilinearly on the pair (v_1, v_2) when the ℓ_j's are fixed – and then we had to verify this this functional as an element of $(V_1 \otimes V_2)^\vee$ depends bilinearly on the pair (ℓ_1, ℓ_2) (in the respective dual modules). This latter verification amounted to certain identities among *linear* functionals on $V_1 \otimes V_2$, and to verify such identities it was sufficient to compare evaluations on members $v_1 \otimes v_2$ of the spanning set of elementary tensors in $V_1 \otimes V_2$. In such cases, the "evaluated" identities boiled down to another property of the "formula" $\ell_1(v_1)\ell_2(v_2) \in F$, namely that it also depends bilinearly on the pair of vectors ℓ_1 and ℓ_2 when the v_j's are fixed. Roughly speaking, the construction of this natural map from the tensor product of dual modules to the dual of the tensor product module comes down to the fact that the expression $\ell_1(v_1)\ell_2(v_2)$ depending on four quantities is linear in any one of them when all others are fixed.

Having made the linear map

$$V_1^\vee \otimes V_2^\vee \to (V_1 \otimes V_2)^\vee$$

we want to show it is an isomorphism. In this case, we will chase bases in a simple manner. Let $\{v_{i,1}\}$ and $\{v_{j,2}\}$ be ordered bases of V_1 and V_2, so $\{v_{i,1} \otimes v_{j,2}\}$ is a basis of $V_1 \otimes V_2$ and $\{v_{i,1}^* \otimes v_{j,2}^*\}$ (using dual basis functionals) is a basis of $V_1^\vee \otimes V_2^\vee$. It suffices to prove that this basis of $V_1^\vee \otimes V_2^\vee$ goes over to the basis of $(V_1 \otimes V_2)^\vee$ that is dual to the basis $\{v_{i,1} \otimes v_{j,2}\}$ of $V_1 \otimes V_2$. Thus, it suffices to show that the functional on $V_1 \otimes V_2$ induced by $v_{i,1}^* \otimes v_{j,2}^*$ sends $v_{r,1} \otimes v_{s,2}$ to 1 if $(r,s) = (i,j)$ and to 0 otherwise. But this is clear: the evaluation of this functional on $v_{r,1} \otimes v_{s,2}$ is $v_{i,1}^*(v_{r,1})v_{j,2}^*(v_{s,2})$, and this is indeed 1 when $r = i$ and $s = j$ and it is 0 otherwise (due to the definition of $\{v_{i,1}^*\}$ and $\{v_{j,2}^*\}$ as dual bases to $\{v_{i,1}\}$ and $\{v_{j,2}\}$ respectively).

3. Trace pairing

We now combine everything we have seen: Hom-tensor, dual-tensor, and commutative-tensor isomorphisms. For finite free F-modules V and V', the Hom-module $\mathrm{Hom}(V, V')$ is also finite free, and it makes sense to consider the composite linear isomorphism

$$\mathrm{Hom}(V,V')^\vee \simeq (V' \otimes V^\vee)^\vee \simeq V'^\vee \otimes V^{\vee\vee} \simeq V'^\vee \otimes V \simeq V \otimes V'^\vee \simeq \mathrm{Hom}(V', V)$$

(using the compatibility of duality with tensor products for finite free modules, which we saw above). This is interesting: we have naturally identified $\mathrm{Hom}(V, V')$ and $\mathrm{Hom}(V', V)$ as dual to each other. That is, if L denotes this composite isomorphism, we have constructed a non-degenerate bilinear form

$$B = B_{V,V'} : \mathrm{Hom}(V', V) \times \mathrm{Hom}(V, V') \to F$$

via $B(T', T) = (L^{-1}(T'))(T) \in F$. In other words, given two linear maps $T' : V' \to V$ and $T : V \to V'$ we have provided a recipe to construct an element $B(T', T) \in F$ in a manner that depends *bilinearly* on the pair T and T'. What could this number be?

We know a couple of ways of extracting scalars from linear maps, such as traces and determinants, but these only apply to self-maps of finite free modules. (Recall from an earlier homework that the theory of trace and determinant on square matrices over any commutative ring is conjugation-invariant, and so is well-defined at the level of endomorphisms of finite free modules over any commutative ring, independent of a choice of basis.) Thus, for example, the self-maps $T' \circ T : V \to V$ and $T \circ T' : V' \to V'$ have traces and determinants. A moment's reflection (check!) shows that $\mathrm{tr}_V(T' \circ T)$ and $\mathrm{tr}_{V'}(T \circ T')$ do depend bilinearly on the pair T and T' (due to the linearity of trace in its argument), whereas such bilinearity fails for the determinant analogues (since determinant does not have good interaction with linear operations in self-maps).

Thus, we are led to guess that perhaps $B(T', T)$ is either $\mathrm{tr}_V(T' \circ T)$ or $\mathrm{tr}_V(T \circ T')$. But which one? Fortunately, these two traces are the same!

Theorem 3.1. *With notation as above,* $B(T', T) = \mathrm{tr}_V(T' \circ T) = \mathrm{tr}_{V'}(T \circ T')$.

Proof. Let us first check that it suffices to prove $B(T', T)$ is equal to one of the two traces. There are two methods: brute force with bases to prove equality of the traces, and thinking to prove that $B(T', T)$ is invariant under swapping (T, T') as well as (V, V'). The brute force method is this: the trace of a square matrix is insensitive to switching the order of multiplication when it is applied to a product of square matrices of the same size, but the same argument works in general. Indeed, for any $n \times n'$ matrix (a_{ij}) and any $n' \times n$ matrix (b_{rs}), the products in both orders are $n \times n$ and $n' \times n'$ matrices whose respective traces can be directly computed to be the same: $\sum_i \sum_j a_{ij} b_{ji} = \sum_r \sum_s b_{rs} a_{sr}$. Here is the method by thinking: inspection of the tensorial construction of B via calculation with elementary tensors from middle terms implies by pure thought that $B_{V,V'}(T',T) = B_{V',V}(T,T')$.

Let us now verify that $B(T',T)$ is equal to $\mathrm{tr}_V(T' \circ T)$. This can certainly be verified directly by picking bases of V and V' and computing everything in terms of matrices relative to these bases and their dual bases (and elementary tensor products thereof). The reader may wish to carry out such a calculation. In what follows we will show an alternative method (that may seem a bit too sneaky, but shows that it *is* possible to pull off the proof with virtually no use of bases): once again we use the principle of chasing elementary tensors out from the middle of a string of isomorphisms.

To prove $B(T',T) = \mathrm{tr}_V(T' \circ T)$, we consider the "left pairing" linear isomorphisms $\mathrm{Hom}(V',V) \simeq \mathrm{Hom}(V,V')^\vee$ given by $T' \mapsto B(T',\cdot)$ and $T' \mapsto \mathrm{tr}_V(T' \circ (\cdot))$. We want to prove that these linear maps agree for all T', and rather than check for all T' we may use the evident *linearity* in T' for both formulas to reduce to checking for T' in a (well-chosen) spanning set of $\mathrm{Hom}(V',V)$. But which spanning set should we use? Well, we pick an elementary tensor $\ell' \otimes v$ in $V'^\vee \otimes V$ and we chase it out to both ends of the long string of isomorphisms: such tensors give rise to elements $T' \in \mathrm{Hom}(V',V)$, and the T' that arise in this way are certainly a spanning set of $\mathrm{Hom}(V',V)$ (why?). Hence, we will check the result for each such T': we will compute T' in terms of ℓ' and v, and we will also compute the linear functional that we get on $\mathrm{Hom}(V,V')$ from $\ell' \otimes v$. We then will check that the functional we obtain is exactly $\mathrm{tr}_V(T' \circ (\cdot))$. Note that to compare two *linear* functionals on $\mathrm{Hom}(V,V')$, we do not need to compare values on all $T \in \mathrm{Hom}(V,V')$, but rather just those from a spanning set: a convenient spanning set is of course the set of T's of the form $\ell(\cdot)v'$ for $\ell \in V^\vee$ and $v' \in V'$ (i.e., those arising from elementary tensors in $V' \otimes V^\vee$).

Now we do the computation. Going to the right from $\ell' \otimes v$, in $\mathrm{Hom}(V',V)$ we get the linear map $T' : v' \mapsto \ell'(v')v$. Going to the left, we get $\ell' \otimes e_v$ in $V'^\vee \otimes V^{\vee\vee}$ (where e_v is the "evaluate on v" linear functional on $V^\vee$), and hence in $(V' \otimes V^\vee)^\vee$ we get the functional $v' \otimes \ell \mapsto e_v(\ell)\ell'(v') = \ell(v)\ell'(v')$. Thus, the functional we get on $\mathrm{Hom}(V,V')^\vee$ sends a linear map $T : V \to V'$ of the form $\ell(\cdot)v'$ (i.e., a T that come from an elementary tensor $v' \otimes \ell \in V' \otimes V^\vee$) to $\ell(v)\ell'(v')$. Our problem is therefore reduced to this: prove that the trace of the composite (the order of composition doesn't matter!) of the linear maps $\ell(\cdot)v'$ (from V to V') and $\ell'(\cdot)v$ (from V' to V) is equal to $\ell(v)\ell'(v')$ for any $v \in V$ and $v' \in V'$.

The composite $(\ell(\cdot)v') \circ (\ell'(\cdot)v)$ from V' to V' sends $v'_1 \in V'$ to $(\ell(\cdot)v')(\ell'(v'_1)v) = \ell(\ell'(v'_1)v)v' = \ell'(v'_1)\ell(v)v' = \ell(v)\ell'(v'_1)v'$. In other words, this is the map $V' \to V'$ that projects onto the span v' with multiplier coefficient function $\ell(v)\ell'$. The trace of this self-map must be proved to equal $\ell(v)\ell'(v')$. The scalar $\ell(v)$ passes through the trace, so we can ignore it: it suffices to prove that the self-map $\ell'(\cdot)v'$ from V' to V' has trace $\ell'(v')$ for each $v' \in V'$.

This proposed identity involves a comparison of quantities that depend *linearly* on v', so it suffices to take v' a member of a basis of V', say an ordered basis whose first member is v'. With respect to this basis, the map sends the first basis vector v' to $\ell'(v')v'$ and sends all other basis vectors to multiples of the first basis vector v'. Hence, the matrix for the map with respect to this basis has upper left entry $\ell'(v')$ and has all other diagonal entries equal to 0. Thus, the trace is $\ell'(v')$ as desired. ■