Math 210A. Tensor algebras, tensor pairings, and duality

Let V be a module over a commutative ring F. We have seen how to make tensor powers $V^{\otimes n}$, symmetric powers $\mathrm{Sym}^n(V)$, and exterior (or "wedge") powers $\wedge^n(V)$ for $n \geq 1$; for $n = 0$ the standard convention is to take these symbols to be F. It has also been seen that these constructions are "natural" in V in the sense that if $T : V \to W$ is a linear map to another F-module then there are unique F-linear maps

$$T^{\otimes n} : V^{\otimes n} \to W^{\otimes n},\ \ \mathrm{Sym}^n(T) : \mathrm{Sym}^n(V) \to \mathrm{Sym}^n(W),\ \ \wedge^n(T) : \wedge^n(V) \to \wedge^n(W)$$

given on elementary tensors (resp. elementary symmetric products, resp. elementary wedge products) by the formulas

$$T^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = T(v_1) \otimes \cdots \otimes T(v_n),\ \ \mathrm{Sym}^n(T)(v_1 \cdot \cdots \cdot v_n) = T(v_1) \cdot \cdots \cdot T(v_n),$$

$$\wedge^n(T)(v_1 \wedge \cdots \wedge v_n) = T(v_1) \wedge \cdots \wedge T(v_n)$$

for $v_i \in V_i$, and moreover these latter operations are compatible with composition in T in an evident manner.

The aim of these notes is to take up a more detailed investigation of how these higher powers of a module interact with bilinear forms and duality in the finite free case, and how we can give the collection of *all* tensor powers (resp. *all* symmetric powers, resp. *all* exterior powers) an interesting multiplicative structure.

Throughout these notes, we fix a coefficient ring F and all modules are understood to be F-modules unless we say otherwise.

1. Pairings of tensor products

We begin with the case of tensor products, as all others will be easily deduced from it after we have done the hard work in this case. Let $V_1, \dots, V_n$ and $W_1, \dots, W_m$ be modules (with $n, m \geq 1$). We would like to construct a bilinear pairing

$$(V_1 \otimes \cdots \otimes V_n) \times (W_1 \otimes \cdots \otimes W_m) \to V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$$

that satisfies the following formula on elementary tensors:

$$(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_m) \mapsto v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

Since such elementary tensors span the respective tensor products of the V_i's and of the W_j's, and a bilinear pairing is uniquely determined by its values on pairs from spanning sets of the respective modules being paired together, such a formula certainly uniquely determines this desired pairing. The problem (as always in the tensor-product business) is therefore one of existence, or really of well-definedness: certainly any elements t and t' in the respective n-fold and m-fold tensor product spaces can be expressed as a finite sum of such respective elementary tensors, and so the value of $t \otimes t'$ is uniquely determined – except that there are usually many ways to write t and t' as sums of elementary tensors and so the difficulty is to ensure that the end result does not depend on the choice of such expressions for t and t'.

To solve our existence problem, if the modules were finite free then one approach (that would have been used in the 19th century) is to choose bases of all V_i's and W_j's to make an unambiguous definition and to then follow the transformation law under change of bases to check that the outputs are independent of these choices (or, in older language, that the coefficient systems of the outputs "transform tensorially" under changes of linear coordinates). This is a rather cumbersome method, and we will avoid it in favor of a more enlightened approach that lets universal mapping properties do all of the work and applies for arbitrary modules (no freeness or finite generation hypotheses). Consider the universal multilinear map

$$\mu : V_1 \times \cdots \times V_n \times W_1 \times \cdots \times W_m \to V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m.$$

Let us *fix* $v_i \in V_i$ for $1 \le i \le n$. Consider the map

$$W_1 \times \ldots W_m \to V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m$$

defined by

$$(w_1, \ldots, w_m) \mapsto v_1 \otimes \ldots v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

This is clearly multilinear in the w_j's, and so by the universal property of $W_1 \otimes \cdots \otimes W_m$ there is a unique linear map

$$m_{v_1,\ldots,v_n} : W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m$$

that satisfies the following formula *on elementary tensors*:

$$m_{v_1,\ldots,v_n}(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \ldots v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

Now comes the key point:

Lemma 1.1. *The element* $m_{v_1,\ldots,v_n} \in \mathrm{Hom}(W_1 \otimes \cdots \otimes W_m, V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m)$ *depends multilinearly on the* v_i*'s.*

Proof. We have to prove that for $a, a' \in F$ and $v_1, v_1' \in V_1$,

$$m_{av_1 + a'v_1', v_2, \ldots, v_n} = am_{v_1,v_2,\ldots,v_n} + a'm_{v_1',v_2,\ldots,v_n}$$

in $\mathrm{Hom}(W_1 \otimes \cdots \otimes W_m, V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m)$, and likewise for linearity in the ith slot for $i > 1$ (and all other slots held fixed). We treat just the case of the first slot, as all others go by the same method.

How are we to verify this proposed equality of maps between two tensor product spaces? It suffices to check equality when the two sides are evaluated on an elementary tensor. Hence, let us compute both sides on $w_1 \otimes \cdots \otimes w_m$ for $w_i \in W_i$. In view of how the linear structure on a Hom-space is *defined*, the desired equality of values is the statement

$$\begin{aligned}(av_1 + a'v_1') \otimes v_2 \otimes \ldots v_n \otimes w_1 \otimes \cdots \otimes w_m = \quad & a(v_1 \otimes v_2 \otimes \ldots v_n \otimes w_1 \otimes \cdots \otimes w_m) \\ & + a'(v_1' \otimes v_2 \otimes \ldots v_n \otimes w_1 \otimes \cdots \otimes w_m)\end{aligned}$$

in $V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$. This equality is simply the multilinearity of the universal map

$$V_1 \times \cdots \times V_n \times W_1 \times \cdots \times W_m \to V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$$

applied in the first slot. ■

In view of the lemma, we have a multilinear map

$$V_1 \times \cdots \times V_n \to \mathrm{Hom}(W_1 \otimes \cdots \otimes W_m, V_1 \otimes \ldots V_n \otimes W_1 \otimes \cdots \otimes W_m)$$

given by $(v_1, \ldots, v_n) \mapsto m_{v_1,\ldots,v_n}$. Hence, by the universal property of the multilinear map

$$V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$$

we obtain a unique linear map

$$L : V_1 \otimes \cdots \otimes V_n \to \mathrm{Hom}(W_1 \otimes \cdots \otimes W_m, V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m)$$

that is given as follows on elementary tensors:

$$L(v_1 \otimes \cdots \otimes v_n) = m_{v_1,\ldots,v_n}.$$

Now observe that for modules U, U', and U'', to give a linear map $T : U \to \mathrm{Hom}(U', U'')$ is the same as to give a bilinear pairing $B : U \times U' \to U''$. Indeed, given T we define B by $(u, u') \mapsto (T(u))(u')$ (which is linear in u' for fixed u because $T(u)$ is linear, and which is linear in u for fixed u' because T is linear), and given B we define T by $T(u) : u' \mapsto B(u, u')$ (this is linear in u' for each $u \in U$ because B is linear in the second variable when the first is fixed, and the resulting association $u \mapsto T(u)$ is linear from U to $\mathrm{Hom}(U', U'')$ because of the definition of the linear structure on $\mathrm{Hom}(U', U'')$ and the linearity of B in the first variable when the second is fixed). One readily checks that these two procedures are inverse to each other. Applying it to our linear map L, we arrive at a bilinear pairing

$$(V_1 \otimes \cdots \otimes V_n) \times (W_1 \otimes \cdots \otimes W_m) \to V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$$

that satisfies

$$(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_m) \mapsto v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

This is exactly the solution to our existence (or well-definedness) problem!

Of course, as with any bilinear pairing, we get a unique factorization through a linear map on the tensor product: there is a linear map

$$(V_1 \otimes \cdots \otimes V_n) \otimes (W_1 \otimes \cdots \otimes W_m) \to V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m \tag{1}$$

that satisfies

$$(v_1 \otimes \cdots \otimes v_n) \otimes (w_1 \otimes \cdots \otimes w_m) \mapsto v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m,$$

and it is unique because $V_1 \otimes \cdots \otimes V_n$ and $W_1 \otimes \cdots \otimes W_m$ are spanned by elementary tensors. By checking with bases of the V_i's and W_j's, one sees that this latter linear map is an *isomorphism*. This expresses an "associativity" property of iterated tensor products, and we leave it to the reader to carry out the same method to prove the existence and uniqueness of a linear isomorphism

$$(V_1 \otimes (V_2 \otimes V_3)) \otimes (V_4 \otimes V_5) \simeq ((V_1 \otimes V_2 \otimes V_3) \otimes V_4) \otimes V_5$$

satisfying $(v_1 \otimes (v_2 \otimes v_3)) \otimes (v_4 \otimes v_5) \mapsto ((v_1 \otimes v_2 \otimes v_3) \otimes v_4) \otimes v_5$; as always, the uniqueness aspect is a trivial consequence of the spanning property of elementary tensors, and it is the existence aspect that requires some thought (but with a bit of experience it becomes mechanical); the basic principle is that when expressions are linear in each variable when all others are held fixed, they give rise to well-defined maps on tensor-product spaces.

Taking $V_i = W_j = V$ for all i and j in what has been done above, we get:

Theorem 1.2. *For any vector space V and $n, m \geq 0$, there is a unique bilinear pairing*

$$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)}$$

satisfying $(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m) \mapsto v_1 \otimes \cdots \otimes v_n \otimes v'_1 \otimes \cdots \otimes v'_m$.

(It is understood that if $n = 0$ or $m = 0$, then these pairings are just pairings against F via scalar multiplication.) The F-bilinear pairing in this theorem is usually denoted $(t, t') \mapsto t \otimes t'$ (by unfortunate but pervasive abuse of notation, justified by the fact that (1) is an isomorphism). The key fact is that this procedure is associative:

Lemma 1.3. *For $n, n', n'' \geq 0$ and $t \in V^{\otimes n}$, $t' \in V^{\otimes n'}$, and $t'' \in V^{\otimes n''}$, we have $t \otimes (t' \otimes t'') = (t \otimes t') \otimes t''$ in $V^{\otimes (n+n'+n'')}$.*

Proof. Since $(t, t') \mapsto t \otimes t'$ is F-bilinear, both sides of the proposed equality inside of $V^{\otimes (n+n'+n'')}$ are trilinear in t, t', and t''. Hence, it suffices to check the equality when each of these three vectors is restricted to lie in spanning sets of the respective spaces $V^{\otimes n}$, $V^{\otimes n'}$, and $V^{\otimes n''}$. The cases when

n, n', or n'' vanish are trivial, so we may assume all three are positive. Naturally enough, we take as spanning sets the elementary tensors in these spaces, and if

$$t = v_1 \otimes \cdots \otimes v_n, \ \ t' = v'_1 \otimes \cdots \otimes v'_{n'}, \ \ t'' = v''_1 \otimes \cdots \otimes v''_{n''}$$

then both sides of our proposed equality are equal to

$$v_1 \otimes \cdots \otimes v_n \otimes v'_1 \otimes \cdots \otimes v'_{n'} \otimes v''_1 \otimes \cdots \otimes v''_{n''} \in V^{\otimes(n+n'+n'')}.$$

■

The associativity in the lemma permits us to make an important construction in algebra:

Definition 1.4. The *tensor algebra* on a module V is the (generally not finitely generated) module

$$\mathrm{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

with multiplication law $m_V : \mathrm{T}(V) \times \mathrm{T}(V) \to \mathrm{T}(V)$ given by

$$m_V((t_n)_{n\geq 0}, (t'_n)_{n\geq 0}) = (\sum_{i+j=n} t_i \otimes t'_j)_{n\geq 0}$$

(where $t_n = 0$ and $t'_n = 0$ for all but finitely many n).

Note that the definition of m_V makes sense because $t_i \otimes t'_j = 0$ for all but finitely many i and j. The associativity lemma for the pairings $V^{\otimes n} \times V^{\otimes n'} \to V^{\otimes(n+n')}$ ensures that m_V is in fact an associative law of composition (check!), thereby permitting us to drop parentheses when iterating m_V, and by definition m_V is clearly distributive over addition.

Example 1.5. Suppose $V = F^d$ with $d > 0$. Let $\{e_i\}$ be the standard basis of V. Let $X_i \in \mathrm{T}(V)$ be the element $(0, e_i, 0, 0, \dots)$. We have $F = V^{\otimes 0}$ in $\mathrm{T}(V)$, and upon unwinding the definitions one sees that elements of $\mathrm{T}(V)$ are "non-commutative polynomials over F" which is to say that this is an associative F-algebra whose elements are all expressible uniquely as F-linear combinations of finite products of "non-commuting variables" $X_1, \dots, X_d$ subject only to the associative law and the condition that everything commutes with elements of F under multiplication. For $d = 3$, a typical element in $\mathrm{T}(V)$ is $aYZ^2XYX + bXY^2ZX + cXYXY + dZ^3X$ with $a, b, c, d \in F$, and this corresponds to

$$(0, 0, 0, ce_1 \otimes e_2 \otimes e_1 \otimes e_2 + de_3 \otimes e_1 \otimes e_1 \otimes e_1, be_1 \otimes e_2 \otimes e_2 \otimes e_3 \otimes e_1, ae_2 \otimes e_3 \otimes e_3 \otimes e_1 \otimes e_2 \otimes e_1, 0, 0, \dots).$$

The strong non-commutativity is due to the fact that $e_i \otimes e_j$ and $e_j \otimes e_i$ are linearly independent in $V^{\otimes 2}$ for $i \neq j$.

Remark 1.6. We also note that iterating these pairings on tensor products link up the higher tensor products with tensor products of two spaces: for $v_1, \dots, v_n \in V = V^{\otimes 1} \subseteq \mathrm{T}(V)$, their product $v_1 v_2 \cdots v_n$ in $\mathrm{T}(V)$ is equal to the universal multilinear value $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ (as it should be!); this is proved by a simple induction on n.

2. Pairings of symmetric and exterior powers

We now seek to prove the existence and uniqueness of bilinear pairings

$$\mathrm{Sym}^n(V) \times \mathrm{Sym}^m(V) \to \mathrm{Sym}^{n+m}(V), \ \ \wedge^n(V) \times \wedge^m(V) \to \wedge^{n+m}(V)$$

satisfying

$$(v_1 \cdot \cdots \cdot v_n, v'_1 \cdot \cdots \cdot v'_m) \mapsto v_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m, \ \ (v_1 \wedge \cdots \wedge v_n, v'_1 \wedge \cdots \wedge v'_m) \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m.$$

Exactly as with the pairings of tensor-product modules, the uniqueness aspect is obvious and it is the existence (or well-definedness) aspect that requires some thought. We will use our results from the above study of tensor products to solve these new existence problems.

Before we explain the argument, we note that the proposed "values" of these pairings on elementary products are multilinear in the v_i's and v'_j's, and are suitably symmetric and alternating as well. This is the basic reason why the well-definedness aspects will work out in the end (just as multilinearity was the reason things worked out in the preceding discussion for pairings of tensor-product modules).

Using the natural linear quotient map from tensor powers onto symmetric and exterior powers, we can use the bilinear tensor pairings that we worked so hard to construct above to make new bilinear pairings:

$$V^{\otimes n} \times V^{\otimes m} \stackrel{\otimes}{\to} V^{\otimes(n+m)} \twoheadrightarrow \mathrm{Sym}^{n+m}(V)$$

and

$$V^{\otimes n} \times V^{\otimes m} \stackrel{\otimes}{\to} V^{\otimes(n+m)} \twoheadrightarrow \wedge^{n+m}(V)$$

satisfying

$$(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m) \mapsto v_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m,$$

$$(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m) \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m$$

respectively. The specification of the values of these bilinear pairings when evaluated on elementary tensors uniquely characterizes these pairings. Our goal is to show that these two pairings between tensor powers uniquely factor through pairings between symmetric and exterior power quotients respectively.

Observe that the quantities

$$v_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m \in \mathrm{Sym}^{n+m}(V), \;\; v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m \in \wedge^{n+m}(V)$$

are respectively symmetric and alternating in the v_i's and v'_j's; that is, the first is invariant under switching two of the vectors and the second vanishes when there is a repetition. In particular, the first expression is insensitive to permutation of the v_i's and also permutation of the v'_j's, whereas the second expression vanishes if $v_i = v_{i'}$ for some $i \neq i'$ or if $v'_j = v'_{j'}$ for some $j \neq j'$. We can therefore apply:

Lemma 2.1. *Let V, V', and V'' be modules and let $B : V^{\otimes n} \times V'^{\otimes n'} \to V''$ be a bilinear pairing.*

1. *If for all $v_1, \dots, v_n \in V$ and $v'_1, \dots, v'_m \in V'$ the value $B(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m) \in V''$ is invariant under swapping $v_i, v_j \in V$ for any $i \neq j$ and is also invariant under swapping $v'_{i'}, v'_{j'} \in V'$ for any $i' \neq j'$ then there is a unique bilinear pairing $\overline{B} : \mathrm{Sym}^n(V) \times \mathrm{Sym}^m(V') \to V''$ such that*
$$\overline{B}(v_1 \cdot \cdots \cdot v_n, v'_1 \cdot \cdots \cdot v'_m) = B(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m)$$
for all $v_1, \dots, v_n \in V$ and $v'_1, \dots, v'_m \in V'$.
2. *If instead $B(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m) \in V''$ vanishes whenever $v_i = v_j$ for some $i \neq j$ or $v'_{i'} = v'_{j'}$ for some $i' \neq j'$ then there is a unique bilinear pairing $\overline{B} : \wedge^n(V) \times \wedge^m(V') \to V''$ such that*
$$\overline{B}(v_1 \wedge \cdots \wedge v_n, v'_1 \wedge \cdots \wedge v'_m) = B(v_1 \otimes \cdots \otimes v_n, v'_1 \otimes \cdots \otimes v'_m)$$
for all $v_1, \dots, v_n \in V$ and $v'_1, \dots, v'_m \in V'$.

Proof. Let us first digress to discussion a general criterion for making bilinear pairings factor through quotients. In general, if $B : W \times W' \to W''$ is a bilinear pairing and $U \subseteq W$ and $U' \subseteq W'$ are submodules such that $B(w, w') = 0$ whenever $w \in U$ *or* $w' \in U'$, then B uniquely factors through a well-defined bilinear pairing $\overline{B} : (W/U) \times (W'/U') \to W''$ given by $\overline{B}(\overline{w}, \overline{w}') = B(w, w')$ where $w \in W$ and $w' \in W'$ are arbitrary choices of representatives of $\overline{w} \in W/U$ and $\overline{w}' \in W/U'$. To see that this makes sense, we simply compute that for $w \in W$, $w' \in W'$, $u \in U$, and $u' \in U'$,

$$B(w + u, w' + u') = B(w, w') + B(w, u') + B(u, w') + B(u, u') = B(w, w')$$

due to the assumption that U is B-perpendicular to everything in W' and that U' is B-perpendicular to everything in W. The bilinearity of $\overline{B}$ is obvious, and the uniqueness aspect for $\overline{B}$ is clear since quotient maps are surjective.

Now returning to the case of interest, we set $W = V^{\otimes n}$, $W' = V^{\otimes m}$, and $W'' = V''$. Two cases of interest are

$$U = \ker(V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(V)),\ \ U' = \ker(V^{\otimes m} \twoheadrightarrow \mathrm{Sym}^m(V))$$

and

$$U = \ker(V^{\otimes n} \twoheadrightarrow \wedge^n(V)),\ \ U' = \ker(V^{\otimes m} \twoheadrightarrow \wedge^m(V)).$$

Under each of the two hypotheses on B we want the corresponding pair U and U' to satisfy the annihilation conditions as in the preceding paragraph. Since B is bilinear, it suffices to check the annihilation conditions using spanning sets of U and U' in each case. In the "symmetric" case a spanning set for U is given by differences

$$v_1 \otimes \cdots \otimes v \otimes \cdots \otimes \widetilde{v} \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes \widetilde{v} \otimes \cdots \otimes v \otimes \cdots \otimes v_n$$

(with all vectors in V and v and $\widetilde{v}$ in the ith and jth slots for some $i \neq j$), and similarly for U' using m replacing n. In the "alternating" case U is spanned by vectors

$$v_1 \otimes \cdots \otimes v \otimes \cdots \otimes v \otimes \dots v_n$$

(with all vectors in V and v in both the ith and jth slots for some $i \neq j$), and U' has a similar spanning set using m replacing n.

The annihilation condition for such spanning sets in each case can be checked against a spanning set in the other slot of the bilinear pairing, such as against the set of all elementary tensors in the other slot. That is, under the symmetry hypothesis we want

$$B(v_1 \otimes \cdots \otimes v \otimes \cdots \otimes \widetilde{v} \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes \widetilde{v} \otimes \cdots \otimes v \otimes \cdots \otimes v_n, t') = 0$$

and

$$B(t, (v'_1 \otimes \cdots \otimes v' \otimes \cdots \otimes \widetilde{v}' \otimes \cdots \otimes v'_m - v'_1 \otimes \cdots \otimes \widetilde{v}' \otimes \cdots \otimes v' \otimes \cdots \otimes v'_m)) = 0$$

for elementary tensors $t \in V^{\otimes n}$ and $t' \in V^{\otimes m}$, and similarly under the alternating hypothesis we want

$$B(v_1 \otimes \cdots \otimes v \otimes \cdots \otimes v \otimes \dots v_n, t') = 0,\ \ B(t, v'_1 \otimes \cdots \otimes v' \otimes \cdots \otimes v' \otimes \dots v'_m) = 0$$

for any elementary tensors $t \in V^{\otimes n}$ and $t' \in V^{\otimes m}$. But (check!) these vanishing statements are exactly the two respective hypotheses imposed on B! ■

By the Lemma, we conclude that there exist bilinear pairings

(2) $$\mathrm{Sym}^n(V) \times \mathrm{Sym}^m(V) \to \mathrm{Sym}^{n+m}(V),\ \ \wedge^n(V) \times \wedge^m(V) \to \wedge^{n+m}(V)$$

given on elementary products by the desired formulas

$$(v_1 \cdot \cdots \cdot v_n, v'_1 \cdot \cdots \cdot v'_m) \mapsto v_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m,\ \ (v_1 \wedge \cdots \wedge v_n, v'_1 \wedge \cdots \wedge v'_m) \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m,$$

and these conditions certainly uniquely determine these bilinear pairings. Note also that as bilinear pairings these even factor through the tensor product of the two factor spaces in each case. That is, we can also say that there exist linear maps

$$\mathrm{Sym}^n(V) \otimes \mathrm{Sym}^m(V) \to \mathrm{Sym}^{n+m}(V), \;\; \wedge^n(V) \otimes \wedge^m(V) \to \wedge^{n+m}(V)$$

respectively satisfying

$$(v_1 \cdot \cdots \cdot v_n) \otimes (v'_1 \cdot \cdots \cdot v'_m) \mapsto v'_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m,$$
$$(v_1 \wedge \cdots \wedge v_n) \otimes (v'_1 \wedge \cdots \wedge v'_m) \mapsto v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m,$$

and such linear maps are uniquely determined by these conditions (as elementary products span each of the spaces in the tensor product). In general, for $s \in \mathrm{Sym}^n(V)$ and $s' \in \mathrm{Sym}^m(V)$ we usually write $s \cdot s'$ to denote the image of (s, s') in $\mathrm{Sym}^{n+m}(V)$ under (2), and for $\omega \in \wedge^n(V)$ and $\omega' \in \wedge^m(V)$ we usually write $\omega \wedge \omega'$ to denote the image of (ω, ω') in $\wedge^{n+m}(V)$ under (2). Exactly as with tensor products in Lemma 1.3, we have an associativity lemma for these new "products":

Lemma 2.2. *For $n, n', n'' \geq 0$ and elements $s \in \mathrm{Sym}^n(V)$, $s' \in \mathrm{Sym}^{n'}(V)$, $s'' \in \mathrm{Sym}^{n''}(V)$ and $\omega \in \wedge^n(V)$, $\omega' \in \wedge^{n'}(V)$, $\omega'' \in \wedge^{n''}(V)$ we have*

$$s \cdot (s' \cdot s'') = (s \cdot s') \cdot s'', \;\; \omega \wedge (\omega' \wedge \omega'') = (\omega \wedge \omega') \wedge \omega''$$

in $\mathrm{Sym}^{n+n'+n''}(V)$ and $\wedge^{n+n'+n''}(V)$ respectively.

Proof. The identities are trilinear, and so it suffices to check on elementary products. This is a simple calculation, exactly as in the proof of Lemma 1.3 (treating the cases $n = 0$ or $n' = 0$ or $n'' = 0$ separately). ■

The interested reader should compare our elegant proof of associativity with the cumbersome method used in many references that develop tensor products "incorrectly", wherein there are painful calculations with lots of intervening factorials (due to using the "wrong" definitions and foundations; see Theorem 3.4*ff.* below for more on this issue).

Whereas there was no issue of sign-commutativity in the case of pairings of tensor powers, for symmetric and exterior powers there are further simple identities related to the possible symmetry of these multiplication laws:

Lemma 2.3. *For $s \in \mathrm{Sym}^n(V)$, $s' \in \mathrm{Sym}^m(V)$, $\omega \in \wedge^n(V)$, and $\omega' \in \wedge^m(V)$ we have*

$$s \cdot s' = s' \cdot s, \;\; \omega \wedge \omega' = (-1)^{nm} \omega' \wedge \omega$$

in $\mathrm{Sym}^{n+m}(V)$ and in $\wedge^{n+m}(V)$ respectively.

Proof. These identities are bilinear and so to verify them it is sufficient to consider elementary products. That is, for $v_1, \dots, v_n, v'_1, \dots, v'_m \in V$ we want

$$v_1 \cdot \cdots \cdot v_n \cdot v'_1 \cdot \cdots \cdot v'_m = v'_1 \cdot \cdots \cdot v'_m \cdot v_1 \cdot \cdots \cdot v_n, \;\; v_1 \wedge \cdots \wedge v_n \wedge v'_1 \wedge \cdots \wedge v'_m = (-1)^{nm} v'_1 \wedge \cdots \wedge v'_m \wedge v_1 \wedge \cdots \wedge v_n$$

in $\mathrm{Sym}^{n+m}(V)$ and in $\wedge^{n+m}(V)$ respectively. Rather more generally, we claim that for any multilinear maps

$$S : V^{n+m} \to W, \;\; A : V^{n+m} \to W'$$

that are respectively symmetric and alternating in all $n + m$ variables in V, we have

$$S(v_1, \dots, v_n, v'_1, \dots, v'_m) = S(v'_1, \dots, v'_m, v_1, \dots, v_n),$$
$$A(v_1, \dots, v_n, v'_1, \dots, v'_m) = (-1)^{nm} A(v'_1, \dots, v'_m, v_1, \dots, v_n).$$

In each case, the ordered lists of $n + m$ variables on the two sides are related by nm swaps of pairs of vectors (first move v_n past all v'_j's, then move v_{n-1} past the v'_j's, and so on down to v_1 moving past all v_j''s). Each of the nm swaps has no impact on the symmetric S, but introduces a sign on the alternating A. ■

Exactly as with the tensor algebra $\mathrm{T}(V)$, we can use these associative pairings to define the associative *symmetric algebra* and *exterior algebra*

$$\mathrm{Sym}(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V), \ \ \wedge(V) = \bigoplus_{n\geq 0} \wedge^n(V).$$

To make these concrete, suppose that V is finite free with an ordered basis $e_1, \dots, e_d$. Let X_i correspond to e_i viewed as an element of $V = \mathrm{Sym}^1(V)$ or $V = \wedge^1(V)$ respectively. The symmetric algebra on V is identified with a commutative polynomial ring in d variables X_j over F. Likewise, the exterior algebra on V is identified with a polynomial ring in d "alternating" variables X_j over F that satisfy $X_j^2 = 0$ and $X_iX_j = -X_jX_i$. By induction on n, we also recover a link between higher symmetric and exterior powers and multiplication in these associative algebras (analogous to Remark 1.6): for $v_1, \dots, v_n \in V = \mathrm{Sym}^1(V)$ the product $v_1 \dots v_n$ in $\mathrm{Sym}(V)$ is the "universal symmetric pairing" $v_1 \cdot \dots \cdot v_n \in \mathrm{Sym}^n(V)$, and likewise viewing $v_i \in \wedge^1(V)$ allows us to define the product $v_1 \dots v_n$ in $\wedge(V)$ and this is equal to the "universal alternating pairing" $v_1 \wedge \dots \wedge v_n \in \wedge^n(V)$.

We conclude with an interesting application of the "wedge product" pairing. This result is usually called the *Künneth formula* (and it is a prototype for an important isomorphism in topology):

Theorem 2.4. *For modules V and W, consider V and W as subspaces of $V \oplus W$ via the natural inclusions $v \mapsto (v, 0)$ and $w \mapsto (0, w)$. The linear maps*

$$\wedge^i(V) \otimes \wedge^j(W) \to \wedge^i(V \oplus W) \otimes \wedge^j(V \oplus W) \xrightarrow{\wedge} \wedge^{i+j}(V \oplus W)$$

define a linear map

$$\bigoplus_{i+j=n} (\wedge^i(V) \otimes \wedge^j(W)) \to \wedge^n(V \oplus W)$$

that is moreover an isomorphism.

Proof. Since each element in $V \oplus W$ is a sum of an element in V and an element in W (i.e., $(v, w) = (v, 0) + (0, w)$), any n-fold wedge product of elements in $V \oplus W$ can be expanded out to a sum of wedge products of i elements of V and j elements of W over all possible decompositions $i + j = n$. This proves surjectivity. Also, if V and W are free modules whose ranks add up to n then the map in question is a surjective map between free modules of rank 1, so it is an isomorphism. We will use this special case in a sneaky way later.

For injectivity in general, we build a left-inverse (which is then also a right-inverse). For $i + j = n$ consider the multilinear map

$$(V \oplus W)^{\times n} \to \wedge^i(V) \otimes \wedge^j(W)$$

defined by

$$((v_1, w_1), \dots, (v_n, w_n)) \mapsto \sum_{I,J} \sigma_{I,J} v_I \otimes w_J$$

where $v_I := v_{m_1} \wedge \dots \wedge v_{m_i}$ for $\{m_1, \dots, m_i\} = I$ with $m_1 < \dots < m_i$ and similarly for w_J, with $\{I, J\}$ a partition of $\{1, \dots, n\}$ and $\sigma_{I,J}$ the sign of the permutation of $\{1, \dots, n\}$ that moves J entirely to the right of I. (The motivation is given by formally expanding out an expression

$$(v_1 + w_1) \wedge \dots \wedge (v_n + w_n)$$

in terms of wedge products of v's and w's, with the v's moved to the left and the w's moved to the right.) This map is visibly multilinear. Provided it is alternating, it uniquely factors through a linear map $\wedge^n(V \oplus W) \to \wedge^i(V) \otimes \wedge^j(W)$ whose composite with the initial linear map (restricted to the (i,j)-component) is the identity map, thereby proving the desired injectivity.

To check the alternating property, we assume $(v_a, w_a) = (v_b, w_b)$ for some $1 \le a \ne b \le n$. If I contains both a and b or J does then v_I or w_J vanish respectively. Suppose instead that I contains a and J contains b. Then there is another term using the partition $\{I', J'\}$ with a and b swapped but everything else left alone. This term has J' containing a and I' containing b. By working with the pairs (I, J) and (I', J') in this way, it suffices to show that the two associated terms $v_I \otimes w_J$ and $v_{I'} \otimes w_{J'}$ add up to zero. Observe that $v_{I'}$ and v_I involve the same v_r's (as $v_a = v_b$) but there is a sign $\epsilon_{I,I'}$ introduced since the vector $v_a = v_b$ has moved within the strictly increasing list of indices. The same happens for comparison of $w_{J'}$ and w_J, with some sign $\epsilon_{J,J'}$. Hence, $\sigma_{I,J} v_I \otimes w_J + \sigma_{I',J'} v_{I'} \otimes w_{J'} = (\sigma_{I,J}\epsilon_{I,I'} + \sigma_{I',J'}\epsilon_{J,J'}) v_I \otimes w_J$, so our task is to prove that the coefficient vanishes in F. It looks like a real pain to keep track of all of the permutation signs involved, so we save effort by observing that this vanishing problem has nothing at all to do with our specific modules! In particular, consider the case that V and W are finite free modules of respective ranks equal to i and j, and choose the v's so that $v_a = v_b$ with $\{v_m\}_{m \in I}$ a basis of V and choose the w's so that $w_a = w_b$ with $\{w_r\}_{r \in J}$ a basis of W. In this case the mystery coefficient is the multiplier against $v_I \otimes w_J$ when computing the preimage of

$$(v_1, w_1) \wedge \cdots \wedge (v_a, w_a) \wedge \cdots \wedge (v_b, w_b) \wedge \cdots \wedge (v_n, w_n) = 0$$

under our initial map of interest that is moreover *known* to be an isomorphism in exactly these special cases. Hence, the coefficient vanishes in F, as required. ∎

For applications in the study of orientations on manifolds and transverse intersections of submanifolds, there are some further results on pairings of top exterior powers that are particularly useful, as we now show.

Theorem 2.5. *Assume F is a field. Let V be a nonzero vector space with dimension n and $W \subseteq V$ a nonzero proper subspace with dimension m. There exists a unique linear map of 1-dimensional vectors spaces $\wedge^m(W) \otimes \wedge^{n-m}(V/W) \to \wedge^n(V)$ satisfying*

$$(w_1 \wedge \cdots \wedge w_m) \otimes (\overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-m}) \mapsto w_1 \wedge \cdots \wedge w_m \wedge v_1 \wedge \cdots \wedge v_{n-m}$$

(for any $v_i \in V$ representing $\overline{v}_i \in V/W$). Moreover, this is an isomorphism.

Proof. By the same principles of chasing multilinear and alternating expressions, for existence and uniqueness it is equivalent to show that the map $W^m \times (V/W)^{n-m} \to \wedge^n(V)$ given by

$$(w_1, \dots, w_m, \overline{v}_1, \dots, \overline{v}_{n-m}) \mapsto w_1 \wedge \cdots \wedge w_m \wedge v_1 \wedge \cdots \wedge v_{n-m}$$

(with $v_i \in V$ representing $\overline{v}_i \in V/W$) is well-defined (i.e., independent of the choices of representatives v_i), multilinear, and alternating in the w_j's for fixed $\overline{v}_i$'s as well as alternating in the $\overline{v}_i$'s for fixed w_j's. Indeed, multilinearity will provide a bilinear pairing $W^{\otimes m} \times (V/W)^{\otimes(n-m)} \to \wedge^n(V)$ given by the desired formula on elementary tensors, and then the alternating properties would allow us to use Lemma 2.1 to get the desired pairing on the exterior-power quotients of these tensor powers.

For well-definedness, we choose $w'_1, \dots, w'_{n-m} \in W$ and we must show that in $\wedge^n(V)$

$$w_1 \wedge \cdots \wedge w_m \wedge (v_1 + w'_1) \wedge \cdots \wedge (v_{n-m} + w'_{n-m}) = w_1 \wedge \cdots \wedge w_m \wedge v_1 \wedge \cdots \wedge v_{n-m}.$$

Using multilinearity to expand out the left side as a sum of elementary n-fold wedge products, each such term involves $m+1$ vectors from the m-dimensional space $W \subseteq V$, so each such term is an n-fold wedge product of a linearly dependent set of vectors in V. Hence, all such terms in $\wedge^n(V)$ vanish. This settles well-definedness.

With well-definedness established, we turn to multilinearity in the w_j's and the $\overline{v}_i$'s. The situation for the w_j's is clear since wedge products are multilinear, and to handle the $\overline{v}_i$'s we simply need to make an artful choice of representatives. More specifically, for scalars $a, a' \in F$ and vectors $v_i, v_i' \in V$ we can use $av_i + a'v_i' \in V$ as a representative for $a\overline{v}_i + a'\overline{v}_i' \in V/W$, and so the multilinearity in the ith entry from V/W is obtained. The alternating property in the w_j's and $\overline{v}_i$'s separately is clear: we have vanishing of the formula when $w_j = w_{j'}$ for some $j \neq j'$, and if $\overline{v}_i = \overline{v}_{i'}$ for some $i \neq i'$ then we can use a common representative $v_i = v_{i'}$ in V for this common vector in V/W.

To check that the unique linear map just constructed is an isomorphism, we consider bases. Let $\{w_j\}$ be a basis of W and let $\{\overline{v}_i\}$ be a basis of V/W. A set of representatives $\{v_i\}$ in V is therefore independent and the collection $\{w_1, \dots, w_m, v_1, \dots, v_{n-m}\}$ is a basis for V. Thus, the vectors

$$w_1 \wedge \cdots \wedge w_m \in \wedge^m(W), \;\; \overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-m} \in \wedge^{n-m}(V/W)$$

are bases in these two lines, and likewise

$$w_1 \wedge \cdots \wedge w_m \wedge v_1 \wedge \cdots \wedge v_{n-m} \in \wedge^n(V)$$

is a basis vector for the line $\wedge^n(V)$. The map we've constructed has the form $T : L \otimes L' \to L''$ where L, L', and L'' are 1-dimensional and we have shown that it satisfies $\ell \otimes \ell' \mapsto \ell''$ where $\ell \in L$, $\ell' \in L'$, and $\ell'' \in L''$ are basis vectors, and so such a map T between 1-dimensional spaces has to be an isomorphism (as it carries the basis vector $\ell \otimes \ell'$ to the basis vector ℓ''). ■

As the proof will show, the following theorem is a mild generalization (in the linear algebra setting) of the Künneth formula isomorphism constructed above.

Theorem 2.6. *Assume F is a field. Let $W_1, \dots, W_N$ be a collection of mutually transverse nonzero proper subspaces of a vector space V of dimension $n > 0$, and let $W' = \cap W_i$, so if $c_i = \mathrm{codim}(W_i)$ then $c = \mathrm{codim}(W')$ is equal to $\sum c_i$. There exists a unique linear map of 1-dimensional spaces $\wedge^{c_1}(V/W_1) \otimes \cdots \otimes \wedge^{c_N}(V/W_N) \to \wedge^c(V/W')$ satisfying*

$$(\overline{v}_{11} \wedge \cdots \wedge \overline{v}_{1,c_1}) \otimes \cdots \otimes (\overline{v}_{N,1} \wedge \cdots \wedge \overline{v}_{N,c_N}) \mapsto v_{11} \wedge \cdots \wedge v_{1,c_1} \wedge \cdots \wedge v_{N,1} \wedge \cdots \wedge v_{N,c_N}$$

for $v_{ij} \in V/W'$ representing $\overline{v}_{ij} \in V/W_i$. Moreover, this map is an isomorphism.

Proof. To prove the existence (including well-definedness) and uniqueness of the linear map given by the proposed formula, we can argue exactly as in the preceding proof, essentially by working with a map

$$(V/W_1)^{c_1} \times \cdots \times (V/W_N)^{c_N} \to \wedge^c(V/W')$$

and checking it is suitably multilinear and alternating (once it is verified to be well-defined). To check the isomorphism aspect, it is convenient to slightly simplify the initial setup. Using the isomorphism $V/W_i \simeq (V/W')/(W_i/W')$, we may replace V and W_i with V/W' and W_i/W' respectively to reduce to the case $W' = 0$. In particular, $\dim V = \mathrm{codim}(W') = \sum c_j$. Hence, the map

$$V \to (V/W_1) \oplus \cdots \oplus (V/W_N)$$

is injective and dimension considerations then force it to be an isomorphism. It follows that we can find a basis of V whose first c_1 vectors reduce to a basis of V/W_1, whose next c_2 vectors reduce to a basis of V/W_2, and so on. Using such a collection of vectors, we see that the map of interest between 1-dimensional spaces carries a basis vector to a basis vector and hence is an isomorphism. ■

3. Bilinear pairings of tensor, symmetric, and exterior powers

Let $B : V \times W \to F$ be a bilinear pairing. We shall now use the preceding considerations to define induced bilinear pairings

$$B^{\otimes n} : V^{\otimes n} \times W^{\otimes n} \to F,\ \ \mathrm{Sym}^n(B) : \mathrm{Sym}^n(V) \times \mathrm{Sym}^n(W) \to F,\ \ \wedge^n(B) : \wedge^n(V) \times \wedge^n(W) \to F.$$

Perhaps the most important B is the evaluation pairing $V \times V^\vee \to F$ for a finite free F-module V, and in this instance a special case of the problem for tensor powers is given in the homework: a natural identification of $(V^\vee)^{\otimes n}$ with the dual of $V^{\otimes n}$. This amounts to giving a natural non-degenerate bilinear pairing $V^{\otimes n} \times (V^\vee)^{\otimes n} \to F$. The interested reader can check that this homework construction is recovered as $B^{\otimes n}$ (to be defined in a moment) when B is the evaluation pairing.

We first propose formulas to uniquely characterize each of the pairings we will construct, namely we specify the pairings between elementary tensor products, symmetric products, and wedge products: for $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$ we wish to require

$$B^{\otimes n}(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n) = \prod_{i=1}^n B(v_i, w_i)$$

and

$$\mathrm{Sym}^n(B)(v_1 \cdot \cdots \cdot v_n, w_1 \cdot \cdots \cdot w_n) = \sum_{\sigma \in S_n} \prod_{i=1}^n B(v_i, w_{\sigma(i)}) = \sum_{\sigma \in S_n} B^{\otimes n}(v_1 \otimes \cdots \otimes v_n, w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}),$$

$$\wedge^n(B)(v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^n B(v_i, w_{\sigma(i)}) = \det(B(v_i, w_j))$$

where S_n denotes the symmetric group on n letters. The only thing that requires an argument is to prove that such formulas are well-posed and more specifically do really arise from bilinear pairings between the desired vector spaces. Lemma 2.1 provides the key: once we make $B^{\otimes n}$, then the existence of the other two pairings follows immediately (check!) from the symmetric and alternating natures of the proposed formulas in these two cases. Hence, our only task is to construct $B^{\otimes n}$. The proposed formula $\prod_{i=1}^n B(v_i, w_i)$ is certainly multilinear in each of the v_i's and w_j's when all other vectors are held fixed, so there is a unique linear map

$$V \otimes \cdots \otimes V \otimes W \otimes \cdots \otimes W \to F$$

(with n copies of V and n copies of W) given by $v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n \mapsto \prod_{i=1}^n B(v_i, w_i)$, so we get $B^{\otimes n}$ by composing this with the bilinear pairing

$$V^{\otimes n} \times W^{\otimes m} \to V \otimes \cdots \otimes V \otimes W \otimes \cdots \otimes W$$

that we constructed with much effort in §1.

Let us now specialize these considerations to the case of a finite free F-module V and the evaluation pairing $B : V \times V^\vee \to F$ given by $B(v, \ell) = \ell(v)$. We get unique bilinear pairings

$$V^{\otimes n} \times (V^\vee)^{\otimes n} \to F,\ \ \mathrm{Sym}^n(V) \times \mathrm{Sym}^n(V^\vee) \to F,\ \ \wedge^n(V) \times \wedge^n(V^\vee) \to F$$

that are respectively characterized by the three formulas

$$(v_1 \otimes \cdots \otimes v_n, \ell_1 \otimes \cdots \otimes \ell_n) \mapsto \prod_{i=1}^n \ell_i(v_i),\ \ (v_1 \cdot \cdots \cdot v_n, \ell_1 \cdot \cdots \cdot \ell_n) \mapsto \sum_{\sigma \in S_n} \prod_{i=1}^n \ell_i(v_{\sigma(i)}),$$

and

$$(v_1 \wedge \cdots \wedge v_n, \ell_1 \wedge \cdots \wedge \ell_n) \mapsto \det(\ell_i(v_j)).$$

Are these perfect pairings? We shall see that the theory for symmetric powers is a little more tricky than for tensor and exterior powers, so we first consider the latter two cases.

Theorem 3.1. *Assume V is a finite free F-module of rank $d > 0$. The preceding bilinear pairings*

$$V^{\otimes n} \times (V^{\vee})^{\otimes n} \to F, \;\; \wedge^n(V) \times \wedge^n(V^{\vee}) \to F$$

are perfect. If $\{e_i\}$ is a basis of V with dual basis $\{e_i^*\}$ in $V^{\vee}$ then the dual basis to $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ is $\{e_{i_1}^* \otimes \cdots \otimes e_{i_n}^*\}$ (with $1 \leq i_j \leq d$) and the dual basis to $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ is $\{e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*\}$ (with $1 \leq i_1 < \cdots < i_n \leq d$).

Proof. It suffices to check that the proposed bases and dual bases are in fact dual with respect to the given pairings, and this is immediate from the given formulas for the pairings on elementary products since $e_j^*(e_i)$ vanishes for $i \neq j$ and equals 1 for $i = j$. ■

Corollary 3.2. *Assume V is a finite free F-module of rank $d > 0$. There are natural isomorphisms $(V^{\vee})^{\otimes n} \simeq (V^{\otimes n})^{\vee}$ and $\wedge^n(V^{\vee}) \simeq (\wedge^n(V))^{\vee}$. Moreover, if $B : V \times W \to F$ is a perfect bilinear pairing then $B^{\otimes n}$ and $\wedge^n(B)$ are perfect bilinear pairings.*

Proof. The aspect concerning general perfect pairings B follows from the case of the evaluation pairing against the dual space because a bilinear pairing B corresponds to a linear map $B' : W \simeq V^{\vee}$ in the sense that composing the evaluation pairing with $1_V \times B'$ recovers B and the perfectness of B is equivalent to B' being an isomorphism. ■

The situation for symmetric powers is a bit more subtle because the pairing between $e_{i_1} \cdot \cdots \cdot e_{i_n}$ and $e_{i_1}^* \cdot \cdots \cdot e_{i_n}^*$ is generally not 1. To make this precise, fix a monotone sequence $I = \{i_1, \ldots, i_n\}$ of integers between 1 and $d = \mathrm{rank}(V)$. Define

$$e_I = e_{i_1} \cdot \cdots \cdot e_{i_n} \in \mathrm{Sym}^n(V), \;\; e_{I^*} = e_{i_1}^* \cdot \cdots \cdot e_{i_n}^* \in \mathrm{Sym}^n(V^{\vee}).$$

We claim that the pairing of e_I and $e_{I'^*}$ is zero when $I' \neq I$ but that the pairing of e_I and e_{I^*} is equal to $m(I) \stackrel{\mathrm{def}}{=} \prod_{j=1}^d m_j(I)!$, where $m_j(I)$ is the number of $1 \leq r \leq n$ such that $i_r = j$ (so $m(I) = 1$ if and only if I is a strictly increasing sequence). Note first of all that j with $m_j(I) = 0$ or 1 do not impact this product, so it is the repetitions among i_r's that are the real issue. For two monotonically increasing sequences of n indices I and I' we compute $\prod_{r=1}^n e_{i'_r}^*(e_{i_{\sigma(r)}})$ vanishes unless $i_{\sigma(r)} = i'_r$ for all r (in which case it equals 1), and the monotonicity condition on the i_r's and the i'_r's implies that this non-vanishing holds if and only if $I' = I$ and for each $1 \leq j \leq d$ the permutation σ individually permutes the set of $m_j(I)$ consecutive indices r such that $i_r = j$. There are $m_j(I)!$ such permutations of the r's with $i_r = j$ when $m_j(I) > 0$, and so there are $m(I) = \prod_{j=1}^d m_j(I)!$ such permutations in S_n in total. This shows that the pairing between e_I and $e_{I'^*}$ vanishes for $I' \neq I$ but that the pairing of e_I and e_{I^*} is $m(I)$, as claimed.

It can certainly happen that $m(I) = 0$ for some I if F is not a $\mathbf{Z}[1/n!]$-algebra, and so some of these pairings then vanish. Thus, the pairing of symmetric powers is *not* perfect in such cases. On the other hand, when F is a $\mathbf{Z}[1/n!]$-algebra then $m(I) \in F^{\times}$ for all I and so the pairing between nth symmetric powers is perfect (though the bases $\{e_I\}$ and $\{e_{I^*}\}$ are *not* dual to each other when $n > 1$ since some $m(I) \neq 1$ in F in such cases).

Remark 3.3. Assuming $n! in F^{\times}$, we can divide through by the $m(I)$'s to arrange that the pairing is a perfect duality between nth symmetric powers of V and $V^{\vee}$ with the e_I's dual to the e_{I^*}'s. However, this is a very bad thing to do, as the resulting pairings will depend very much on the choice of basis $\{e_i\}$ of V (test $V = F^2$ for yourself).

Note that without restriction on F, but assume our modules to be finite free, our natural pairing of nth symmetric powers of V and $V^\vee$ defines a natural linear map

$$\mathrm{Sym}^n(V^\vee) \to (\mathrm{Sym}^n(V))^\vee, \tag{3}$$

and (if $V \neq 0$) this is an isomorphism if and only if $n! in F^\times$. Likewise, by the method of proof of Corollary 3.2 (reducing to the case of the evaluation pairing), if $V, W \neq 0$ (and are finite free) and $B : V \times W \to F$ is a perfect bilinear pairing then $\mathrm{Sym}^n(B)$ is a perfect pairing if and only if $n! \neq 0$ in F.

We have now reached the point where we can explain what is "wrong" with the development of tensor and exterior algebra in books of Munkres, Hoffman–Kunze, Spivak, and most others. In these books, say with F and field and all vector spaces finite-dimensional, one finds that $V \otimes W$ is defined as the space of bilinear pairings $V \times W \to F$, which is to say that they define it to be what we call $(V \otimes W)^\vee \simeq V^\vee \otimes W^\vee$. Right away we see that this is very bad: there is a confusion between a vector space and its dual, and all "naturality" results will involve maps going in the wrong direction. Let us now discover why this forces such books to introduce weird factorials in the definition of symmetric and wedge products, and why they wind up defining symmetric and exterior powers as *subspaces* of tensor powers instead of as quotients (as should be done). These problems are all introduced by the hidden dual operation just mentioned, due to:

Theorem 3.4. *Let F be a commutative ring and V a finite free F-module. Dualize the surjective quotient maps $V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(V)$ and $V^{\otimes n} \twoheadrightarrow \wedge^n(V)$ to get maps (using (3) and Corollary 3.2)*

$$\mathrm{Sym}^n(V^\vee) \to (\mathrm{Sym}^n(V))^\vee \hookrightarrow (V^{\otimes n})^\vee \simeq (V^\vee)^{\otimes n}$$

and

$$\wedge^n(V^\vee) \simeq (\wedge^n(V))^\vee \hookrightarrow (V^{\otimes n})^\vee \simeq (V^\vee)^{\otimes n}.$$

For $\ell_1, \dots, \ell_n \in V^\vee$, these composite maps satisfy

$$\ell_1 \cdot \dots \cdot \ell_n \mapsto \sum_{\sigma \in S_n} \ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(n)}, \quad \ell_1 \wedge \dots \wedge \ell_n \mapsto \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(n)}.$$

Proof. This is an instructive exercise in unwinding the definitions of the maps involved. ∎

Since we know that the symmetric and exterior powers are naturally quotients of tensor powers, the composite maps in Theorem 3.4 give rise to self-maps

$$\mathrm{Sym}^n(V^\vee) \to (V^\vee)^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(V^\vee), \quad \wedge^n(V^\vee) \to (V^\vee)^{\otimes n} \twoheadrightarrow \wedge^n(V^\vee).$$

It follows immediately from the formulas in Theorem 3.4 that these self-maps of the nth symmetric and exterior powers of $V^\vee$ are *not* the identity but rather are multiplication by the cardinality of S_n, which is to say $n!$. This is why books with the wrong approach to tensor algebra are forced to define symmetric and wedge powers only when $n! \in F^\times$ (usually $F = \mathbf{R}$), and using the "definitions" given by the sums in Theorem 3.4 divided by $n!$: it is only with such division that they are computing the *correct* products in the correct symmetric and exterior powers of the *wrong* space (namely, $V^\vee$), and without such factorials the associativity of symmetric and exterior multiplication as in Lemma 2.2 (with $V^\vee$ secretly replacing V) would break down (just as associativity of multiplication in $\mathbf{R}$ would break down if we tried to redefine xy by multiplying by an extra factor of 7).