Math 210A. Euler characteristic

Let $M^\bullet$ be a finite complex of finite-dimensional vector spaces over a field k. In particular, not only are all M^i's of finite dimension, but so are the homologies $\mathrm{H}^i(M^\bullet)$ for all i. A basic observation of Euler is the identity

$$\sum (-1)^i \dim M^i = \sum (-1)^i \dim \mathrm{H}^i(M^\bullet).$$

In this handout we prove Euler's identity.

The key point is to "chop up" the given complex into many short exact sequences, and repeatedly apply the basic formula

$$\dim V = \dim V' + \dim V''$$

for any short exact sequence $0 \to V' \to V \to V'' \to 0$ of finite-dimensional k-vector spaces. More specifically, using the short sequences

$$0 \to \ker d^i \to M^i \to \mathrm{im}(d^i) \to 0, \;\; 0 \to \mathrm{im}(d^{i-1}) \to \ker d^i \to \mathrm{H}^i(M^\bullet) \to 0$$

we obtain

$$\dim M^i = \dim \ker(d^i) + \dim \mathrm{im}(d^i), \;\; \dim \mathrm{H}^i(M^\bullet) = \dim \ker(d^i) - \dim \mathrm{im}(d^{i-1}),$$

so

$$\dim M^i - \dim \mathrm{H}^i(M^\bullet) = \dim \mathrm{im}(d^i) + \dim \mathrm{im}(d^{i-1}).$$

Multiplying through by $(-1)^i$ and summing over all i, the right side sums to

$$\sum (-1)^i \dim \mathrm{im}(d^i) - (-1)^{i-1} \dim \mathrm{im}(d^{i-1}) = 0.$$

This yields the desired identity.