Math 210A. Duality for finite abelian groups

Let G be a finite abelian group, and k an algebraically closed field with $\mathrm{char}(k) \nmid \#G$. The traditional case is $k = \mathbf{C}$. With k understood (and fixed), we define the *dual group*

$$\widehat{G} = \mathrm{Hom}(G, k^\times).$$

This is a group via pointwise operations, so it is clearly abelian. If n kills G (e.g., $n = \#G$) then every homomorphism $\chi : G \to k^\times$ lands inside of the finite abelian group $\mu_n(k)$ of nth roots of unity in k. Hence, $\widehat{G} = \mathrm{Hom}(G, \mu_n(k))$. This is visibly a finite group. In this handout, we show that the relationship between G and $\widehat{G}$ is very symmetric, akin to that between a finite-dimensional vector space and its linear dual: we will show that $\widehat{G}$ is non-canonically isomorphic to G, and that the natural "double duality" map $G \to (\widehat{G})^\wedge = \mathrm{Hom}(\widehat{G}, k^\times)$ is an isomorphism. (Note that this double-duality map *is* a homomorphism; the argument is similar to the proof that the "double-duality" map for a vector space is linear, and it is left to the reader to verify.)

First we check that non-canonically $\widehat{G} \simeq G$. If $G = G' \times G''$ then

$$\widehat{G} = \mathrm{Hom}(G' \times G'', k^\times) = \mathrm{Hom}(G', k^\times) \times \mathrm{Hom}(G'', k^\times) = \widehat{G'} \times \widehat{G''}.$$

Hence, to prove the non-canonical isomorphism result for G it suffices to do the same for G' and G''. In view of the structure theorem for finite abelian groups, this reduces us to the case when G is cyclic, say of size $d \geq 1$ (with $\mathrm{char}(k) \nmid d$). That is, we can assume $G = \mathbf{Z}/d\mathbf{Z}$. But then $\mathrm{Hom}(G, M)$ is the d-torsion in M for any abelian group M (why?), so taking $M = k^\times$ gives $\widehat{G} = \mu_d(k)$. This is the set of dth roots of unity in k, which is to say roots of the polynomial $X^d - 1$ that is *separable* of degree d. There are d such roots, since k is algebraically closed.

To complete the proof that $\widehat{G}$ is non-canonically isomorphic to G, our task is to prove that $\mu_d(k)$ is cyclic. If $k = \mathbf{C}$ this is geometrically clear from the visualization of dth roots of unity as vertices of a regular d-gon inscribed in the unit circle. For general k, we proceed algebraically as follows. If H is any finite abelian group, by the structure theorem for such groups we see that if H is not cyclic then for some prime ℓ there is a subgroup of the form $(\mathbf{Z}/\ell\mathbf{Z})^2$. In particular, the subgroup $H[\ell]$ of ℓ-torsion in H has size at least $\ell^2 > \ell$. But for $H = \mu_d(k)$ and any prime $\ell | d$, $H[\ell]$ is contained in the group $\mu_\ell(k)$ that has size ℓ. Hence, we have reached a contradiction, so $\mu_d(k)$ is cyclic.

Finally, we establish the "double duality" isomorphism. In view of the non-canonical isomorphism between a finite abelian group and its dual, we see that $\#G = \#\widehat{G}$ for any G. Hence, G and its double-dual have the same size. Thus, to prove that the natural double-duality homomorphism $G \to (\widehat{G})^\wedge$ is an isomorphism it suffices to prove injectivity. In other words, if $g \in G$ and $\chi(g) = 1$ for all $\chi \in \widehat{G}$ then we claim $g = 1$. Equivalently, if $g \neq 1$ then we need to construct *some* $\chi \in \widehat{G}$ such that $\chi(g) \neq 1$. Consider a decomposition $G \simeq \prod C_i$ with cyclic C_i. Since $g \neq 1$, it has nontrivial projection $g_i \in C_i$ for some i. If we can handle the cyclic case then there is a character $\psi : C_i \to k^\times$ such that $\psi(g_i) \neq 1$, so composing ψ with the projection $G \twoheadrightarrow C_i$ defines the desired χ. Hence, we may assume G is cyclic, say $G = \mathbf{Z}/n\mathbf{Z}$ with $\mathrm{char}(k) \nmid n$. But $\mu_n(k)$ is cyclic of size n, so we can choose an isomorphism of groups $\chi : G \simeq \mu_n(k)$. This satisfies $\chi(g) \neq 1$ for all nontrivial $g \in G$.

Remark 0.1. By definition, $G \rightsquigarrow \widehat{G}$ carries short exact sequences to left-exact sequence of finite abelian groups. But by counting sizes, we see that the resulting left-exact sequences are actually *exact*, so dualizing is an exact functor. Explicitly, if $H \subseteq G$ is a subgroup then $(G/H)^\wedge$ is identified with the group $\ker(\widehat{G} \to \widehat{H})$ of characters χ of G such that $\chi|_H = 1$.