Math 154. Discriminant of composite fields

Let F be a number field which is a compositum (over $\mathbf{Q}$) of two subfields K and K'. We assume that $[F:\mathbf{Q}] = [K:\mathbf{Q}][K':\mathbf{Q}]$, or equivalently that $[F:K] = [K':\mathbf{Q}]$ (or equivalently $[F:K'] = [K:\mathbf{Q}]$). Under this assumption, we saw in class that when $\gcd(\mathrm{disc}(K), \mathrm{disc}(K')) = 1$ then the inclusion $\mathscr{O}_K\mathscr{O}_{K'} \subseteq \mathscr{O}_F$ is an equality. In this handout we wish to address a further claim made in class concerning this situation: that $\mathrm{disc}(F)$ can be computed by a universal formula in terms of $\mathrm{disc}(K)$ and $\mathrm{disc}(K')$, or more precisely that

$$\mathrm{disc}(F) = \mathrm{disc}(K)^{[K':\mathbf{Q}]}\mathrm{disc}(K')^{[K:\mathbf{Q}]}.$$

We must emphasize at that outset that such a formula depends crucially on the property that $\mathscr{O}_F = \mathscr{O}_K\mathscr{O}_{K'}$ (which in turn we proved using the hypothesis that $\mathrm{disc}(K)$ and $\mathrm{disc}(K')$ are relatively prime, and we saw that such an equality can fail in the absence of the relative primality of $\mathrm{disc}(K)$ and $\mathrm{disc}(K')$). In more general situations where there is either "loss of field degree" (i.e., $[F:\mathbf{Q}] < [K:\mathbf{Q}][K':\mathbf{Q}]$) or $\gcd(\mathrm{disc}(K), \mathrm{disc}(K')) > 1$ this formula relating the three discriminants will fail. It is generally hopeless to expect to compute $\mathrm{disc}(F)$ just from knowledge of $\mathrm{disc}(K)$, $\mathrm{disc}(K')$, and some field degrees when $\mathrm{disc}(K)$ and $\mathrm{disc}(K')$ share a nontrivial common factor. The explanation for why this is the case rests on a much deeper understanding of the discriminants by means of representation theory, a topic that lies way beyond the level of this course.

To ease the notation, let $n = [K:\mathbf{Q}]$ and $n' = [K':\mathbf{Q}]$, so $[F:\mathbf{Q}] = nn'$. Pick ordered $\mathbf{Z}$-bases $\{e_1, \dots, e_n\}$ of $\mathscr{O}_K$ over $\mathbf{Z}$ and $\{e'_1, \dots, e'_{n'}\}$ of $\mathscr{O}_{K'}$ over $\mathbf{Z}$. Thus, $\mathscr{O}_F = \mathscr{O}_K\mathscr{O}_{K'}$ is spanned over $\mathbf{Z}$ by the pairwise products $e_ie'_{i'}$, and this set of nn' elements of F is therefore a spanning set of F over $\mathbf{Q}$ (as every element of F has the form α/c with $\alpha \in \mathscr{O}_F$ and $c \in \mathbf{Z} - \{0\}$). But $[F:\mathbf{Q}] = nn'$, so this spanning set is also linearly independent over $\mathbf{Q}$. In particular, it is linearly independent over $\mathbf{Z}$, so $\{e_ie'_{i'}\}$ is a $\mathbf{Z}$-basis of $\mathscr{O}_F$. Hence, it may be used to compute $\mathrm{disc}(F)$. More specifically, upon choosing an ordering of the finite set of ordered pairs (i, i') with $1 \le i \le n$ and $1 \le i' \le n'$, the desired relation among discriminants may be expressed as the following identity among determinants:

$$\det(\mathrm{Tr}_{F/\mathbf{Q}}(e_ie'_{i'}e_je'_{j'})_{(i,i'),(j,j')}) \stackrel{?}{=} \det(\mathrm{Tr}_{K/\mathbf{Q}}(e_ie_j))^{n'}\det(\mathrm{Tr}_{K'/\mathbf{Q}}(e'_{i'}e'_{j'}))^n.$$

To convert this into a more tractable form, let us manipulate the traces on the left side. By transitivity of the trace map, we have

$$\mathrm{Tr}_{F/\mathbf{Q}}(e_ie'_{i'}e_je'_{j'}) = \mathrm{Tr}_{F/\mathbf{Q}}((e_ie_j)(e'_{i'}e'_{j'})) = \mathrm{Tr}_{K/\mathbf{Q}}(\mathrm{Tr}_{F/K}((e_ie_j)(e'_{i'}e'_{j'}))) = \mathrm{Tr}_{K/\mathbf{Q}}(e_ie_j \cdot \mathrm{Tr}_{F/K}(e'_{i'}e'_{j'}))$$

since $e_ie_j \in K$. But we saw in class that, due to the field-degree hypothesis $[F:\mathbf{Q}] = [K:\mathbf{Q}][K':\mathbf{Q}]$ (which ensures that a $\mathbf{Q}$-basis of K' may be used as a K-basis of F), for any $c' \in K'$ we have $\mathrm{Tr}_{F/K}(c') = \mathrm{Tr}_{K'/\mathbf{Q}}(c')$. Thus, taking $c' = e'_{i'}e'_{j'}$ for varying (i', j') gives

$$\mathrm{Tr}_{F/\mathbf{Q}}(e_ie'_{i'}e_je'_{j'}) = \mathrm{Tr}_{K/\mathbf{Q}}(e_ie_j\mathrm{Tr}_{K'/\mathbf{Q}}(e'_{i'}e'_{j'})) = \mathrm{Tr}_{K/\mathbf{Q}}(e_ie_j) \cdot \mathrm{Tr}_{K'/\mathbf{Q}}(e'_{i'}e'_{j'}).$$

Now letting $a_{ij} = \mathrm{Tr}_{K/\mathbf{Q}}(e_ie_j)$ and $a'_{i'j'} = \mathrm{Tr}_{K'/\mathbf{Q}}(e'_{i'}e'_{j'})$, our desired identity can be rewritten in the form

$$\det((a_{ij}a'_{i'j'})_{(i,i'),(j,j')}) \stackrel{?}{=} \det(a_{ij})^{n'}\det(a'_{i'j'})^n,$$

where we have chosen an arbitrary but fixed ordering of the finite set of ordered pairs (r, r') with $1 \le r \le n$ and $1 \le r' \le n'$ so as to make sense of the determinant on the left side. In this form we claim such an identity holds with entries in an arbitrary field. This is an exercise on HW4!