1. If $X$ and $Y$ are $C^\infty$ manifolds, show that a continuous map $f : X \to Y$ between the underlying topological spaces is a $C^\infty$ map if for all open $U \subset Y$ and $\varphi \in C^\infty(U)$ the pullback function $\varphi \circ f : f^{-1}(U) \to \mathbb{R}$ lies in $C^\infty(f^{-1}(U))$.

2. (gluing spaces) Let $\{X_i\}$ be a set of topological spaces. Suppose we are given open sets $X_{ij} \subseteq X_i$ for all $i, j$ with $X_{ii} = X_i$ and homeomorphisms $f_{ij} : X_{ij} \simeq X_{ji}$ for all $i, j$ such that $f_{ii}$ is the identity and

$$f_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ij}, \quad f_{jk}(X_{jk} \cap X_{ji}) \circ f_{ij}|_{X_{ij} \cap X_{ik}} = f_{ik}|_{X_{ik} \cap X_{ij}}$$

for all $i, j, k$. We want to glue the $X_i$’s along the $f_{ij}$’s (converting $X_{ij}, X_{ji}$ into $X_i \cap X_j$). Draw pictures to illustrate your arguments below.

(i) Define $X = (\coprod X_i)/\sim$, where $x_i \sim f_{ij}(x_i)$ for $x_i \in X_{ij}$. Show that the natural maps $f_i : X_i \to X$ are injective, with $f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ f_{ij}$, $f_i(X_{ij}) = f_j(X_{ji}) = f_i(X_i) \cap f_j(X_j)$. Thus, we may view $X_i$ as a subset of $X$ and as such $X_{ij} = X_{ji}$ is identified with the overlap $X_i \cap X_j$ inside of $X$ (the identification being $f_{ij}$).

(ii) We define $U \subseteq X$ to be open if and only if $f_i^{-1}(U)$ is open in $X_i$ for all $i$. Show that this makes $X$ a topological space with respect to which $f_i(X_i)$ is open and $f_i$ is a homomorphism of $X_i$ onto $f_i(X_i)$. Thus, we may view $X_i$ as an open subset of $X$. We call $(X, \{f_i\})$ the gluing of the $X_i$’s along the $f_{ij}$’s. Prove the universal property that for any topological space $Y$ and continuous maps $g_i : X_i \to Y$ that “agree on overlaps” in the sense that $g_i|_{X_{ij}} = g_j|_{X_{ji}} \circ f_{ij}$ for all $i, j$, there is a unique continuous map $g : X \to Y$ such that $g \circ f_i = g_i$ for all $i$.

3. (gluing sheaves) Let $X$ be a topological space with an open covering $\{X_i\}$ and let $\mathcal{O}_i$ be a sheaf of $F$-valued functions on $X_i$ for a field $F$. We define $X_{i_1 \ldots i_n} = X_{i_1} \cap \ldots \cap X_{i_n}$. Suppose that for all $i, j$ we have $\mathcal{O}_i(U) = \mathcal{O}_j(U)$ for all open $U \subset X_{ij}$. For every open $U \subseteq X$, define

$$\mathcal{O}(U) = \{(s_i) \in \prod \mathcal{O}(U \cap X_i) | s_i|_{U \cap X_{ij}} = s_j|_{U \cap X_{ji}} \text{ for all } i, j\}.$$

(i) Prove that $\mathcal{O}$ is of local nature. We call it the gluing of the $\mathcal{O}_i$’s.

(ii) For any ringed space $(X, \mathcal{O})$ and open subset $V \subset X$, we define the sheaf of functions $\mathcal{O}|_V$ on $V$ by the rule $U \mapsto \mathcal{O}(U)$ for open $U \subset V$. For a field $F$, formulate what it means to glue a set of ringed spaces $\{(X_i, \mathcal{O}_i)\}$ over $F$ along isomorphisms akin to Exercise 2 between open subspaces $X_{ij} \subset X_i$ (equipped with $\mathcal{O}_i|_{X_{ij}}$). and prove the corresponding universal property within the framework of ringed spaces over $F$. In particular, show that if $(X, \mathcal{O}_X)$ is a ringed space over $F$ and $\{U_i\}$ is an open covering of $X$, then to give a map from $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is “the same” as to give maps $f_i : (U_i, \mathcal{O}_X|_{U_i}) \to (Y, \mathcal{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j$.

4. For any finitely generated reduced $k$-algebra $A$, let $\text{MaxSpec}(A) = (\text{Max}(A), \mathcal{O}_A)$. We define affine $n$-space over $k$ to be $A^n_k = \text{MaxSpec}(k[t_1, \ldots, t_n])$ ($k^n$ equipped with the “sheaf of regular functions” on varying Zariski-open subsets of $k^n$).

(i) Show that $\mathcal{O}_{A^n_k}(A^n_k - \{0\}) = k[x, 1/x]$ and that if $n \geq 2$ then the natural restriction map

$$k[t_1, \ldots, t_n] = \mathcal{O}_{A^n_k}(A^n_k) \to \mathcal{O}_{A^n_k}(A^n_k - \{0\})$$

is an isomorphism. (Hint: for $X = A^n_k$, $X - \{0\}$ is covered by the non-vanishing loci $X_{t_1}, \ldots, X_{t_n}$.)

(ii) Using (i), show that if $n \geq 2$ then $A^n_k - \{0\}$ as a ringed space over $k$ is not isomorphic to $\text{MaxSpec}(A)$ for any finitely generated reduced $k$-algebra $A$. (Hint: if it were, show via (i) that pullback along the natural inclusion $j : A^n_k - \{0\} \to A^n_k$ would have to induce an isomorphism $k[t_1, \ldots, t_n] \simeq A$.) Deduce a contradiction since $j$ is not an isomorphism.) This is the precise sense in which “$A^n_k - \{0\}$ is not affine” when $n \geq 2$. 

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