Math 145. Resolution of singularities for curves

Let k be an algebraically closed field, and C an irreducible 1-dimensional smooth variety (hence quasi-compact, separated). I stated in class that there exists an open immersion $C \hookrightarrow \overline{C}$ into an irreducible *smooth projective* curve, so any such C is automatically quasi-projective, but for the proof I only had time to reduce to the case where C is affine (though this reduction step is where we needed the condition that C be separated). The purpose of this handout is to complete the argument by handling the case of affine C. I must again emphasize that the crucial point is that we can find an open immersion of C into a projective irreducible curve $\overline{C}$ which is *smooth*. Indeed, for an irreducible affine curve C, say in $\mathbf{A}^n$, its closure in $\mathbf{P}^n$ is certainly an irreducible projective curve, but probably isn't smooth! For example, consider (in characteristic distinct from 2) any curve defined by $y^2 = f(x)$ where f has distinct roots but degree larger than 3. So finding the smooth projective curve requires some thought.

I should mention at the start that it is actually true that any abstract irreducible 1-dimensional variety is quasi-projective, with no smoothness hypotheses at all, but this is beyond the level of the course. Once we know that irreducible 1-dimensional smooth varieties are automatically quasi-projective, it is reasonable to ask whether there is an intrinsic characterization of which ones are actually projective (i.e., admit a *closed* immersion into a projective space). Again, there is a very satisfactory answer to this question but in order to formulate it in the most natural context one needs some more advanced ideas (related to abstracting the notion of compactness in ordinary topology into a form more suitable for use in algebraic geometry, somewhat in the same spirit as we abstracted the Hausdorff condition into the concept of separatedness). What we will be able to prove later, with the help of the Riemann-Roch Theorem, is that 1-dimensional irreducible smooth varieties are either projective or *affine* (but this is not a satisfactory answer to the "intrinsic characterization" question, because it does not generalize at all to higher dimensions).

The fact that *projective smooth* curves are uniquely determined up to canonical isomorphism by their function fields thereby ensures, by the above results, that *any* 1-dimensional variety C is birational to a "unique" irreducible smooth projective curve (namely, the one containing the normalization of C as a dense open). But *any* finitely generated transcendence degree 1 extension K of k does have the form $K \simeq k(C)$ for an irreducible affine curve C (e.g., K is the fraction field of a domain A finitely generated over k, and from our earlier work relating transcendence degree and dimension we know that $C = \mathrm{MaxSpec}(A)$ is an irreducible affine variety of dimension 1 with function field K). We thereby conclude that the category of finitely generated transcendence degree 1 extensions of k, equipped with k-algebra maps as the morphisms, is "the same" as the category of irreducible *smooth projective* curves over k, with non-constant maps (or equivalently, dominant rational maps) as the morphisms. This classical dictionary between fields and curves is basic in the study of curves and fails miserably in higher dimensions (hence the reason why algebraic curves are much easier to study than higher-dimensional things).

Before we address the matter of finding a projective smooth curve containing a given affine one as a dense open, we need to prove a general lemma that I used in class in the 1-dimensional case:

Lemma 0.1. *Let $f : X \to Y$ be an affine map between abstract algebraic sets. If Y is separated, so is X.*

I stated this lemma in class for finite maps, because that was the only case we needed. But the proof only requires that f be an affine map (which finite maps certainly are).

Proof. We need to show that the diagonal map $\Delta_X : X \to X \times X$ is a closed immersion. Since Y is separated, the graph map $\Gamma_f : X \to X \times Y$ is a closed immersion. The preimage Z of $\Gamma_f(X)$ under

$$1 \times f : X \times X \to X \times Y$$

is therefore a closed subset of $X \times X$, and it is exactly $\{(x, x') \in X \times X \mid f(x) = f(x')\}$. This *closed* subset Z in $X \times X$ certainly contains $\Delta_X(X)$, so to show that Δ_X is a closed immersion, we can work locally around points $z \in Z$ (this follows from the *method* of proof of the lemma in the handout on graphs). Choose a point $z = (x, x') \in Z$, so $f(x) = f(x') = y \in Y$. Let U be an open affine in Y around y, so by *affineness* of the map f, $V = f^{-1}(U)$ is an open affine in X and contains x, x'. Thus, $V \times V$ is an open in $X \times X$ around our chosen point z in the closed set Z. Hence, as we noted above (using the lemma in the handout on graphs), it suffices to show that $\Delta_X^{-1}(V \times V) \to V \times V$ is a closed immersion for all such V. But it is obvious that the open set $\Delta_X^{-1}(V \times V)$ in X is exactly V, so we are left with studying the diagonal map of V. But V is affine, hence separated, so its diagonal map is a closed immersion, as desired.

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With this technical point from lecture settled, we now proceed to complete the proof of the main fact from class.

Theorem 0.2. *Let C be an irreducible 1-dimensional smooth variety. There exists an irreducible smooth projective curve $\overline{C}$ and an open immersion $C \hookrightarrow \overline{C}$.*

Proof. In class we saw how the separatedness of C, together with induction on the size of an open affine covering of C (quasi-compactness!), permits us to reduce the general case to the case of affine C. Thus, we now assume C is affine. By Noether normalization, there exists a finite surjective map $f : C \to \mathbf{A}^1$. *This is where we use that C is affine.* In particular, we have a dominant map $C \to \mathbf{P}^1$, corresponding to a *finite* extension of function fields $k(t) \hookrightarrow k(C)$; here we identify $\mathbf{A}^1 = \mathbf{P}^1 - \{[1, 0]\}$.

On an earlier homework, you developed a theory of normalization. There you showed that if X is an irreducible abstract algebraic set, then there exists a normal irreducible abstract algebraic set $\widetilde{X}$ equipped with a finite map $\widetilde{X} \to X$ which is an isomorphism over a dense open, and such that for *any* normal irreducible abstract algebraic set Y and any dominant map $Y \to X$, there is a unique factorization through $\widetilde{X} \to X$ (and moreover $Y \to \widetilde{X}$ is dominant). Ultimately, this came from the algebraic fact that if A is a domain with integral closure $\widetilde{A}$ in its fraction field K, then any *injection* $A \hookrightarrow B$ into an integrally closed domain B with fraction field L (naturally viewed as an extension of K) automatically has B containing $\widetilde{A}$ as a subring of L; that is, the map $A \hookrightarrow B$ factors as

$$A \hookrightarrow \widetilde{A} \hookrightarrow B.$$

The geometric theory of normalization was just a globalization of this affine fact.

What makes the global construction of normalizations work is the affine fact that if A is a domain with fraction field K, then the formation of the integral closure $\widetilde{A}$ of A in K is compatible with localization on A: i.e., $(\widetilde{A})_a = \widetilde{A_a}$ in K for non-zero $a \in A$. But in fact if L/K is any finite extension field at all, then the formation of the integral closure of A in L is likewise compatible with localizing on A (proof?). Using this fact, we can similarly form "normalizations" of any X in finite extensions of the function field of X. Before we make this precise, we give the algebraic interpretation. Algebraically, this corresponds to the following generalization of the algebraic assertion at the end of the preceding paragraph: if K'/K is a fixed finite extension and $\widetilde{A}'$ is the integral closure of A in K' (so this domain has fraction field K': why?), then for any injection $A \hookrightarrow B$ into a normal domain B with fraction field L and any embedding $K' \hookrightarrow L$ *over* K, the ring B automatically contains the subring $\widetilde{A}'$. Recalling the basic finiteness results for integral closures of finitely generated domains over an algebraically closed field, we see that if A is a domain finitely generated over k, then the k-algebra domain $\widetilde{A}'$ introduced above is also *finitely generated* over k (so the affine variety $\mathrm{MaxSpec}(\widetilde{A}')$ makes sense, and of course has the same dimension as the affine variety $\mathrm{MaxSpec}(A)$ by transcendence degree considerations for its function field K' over k.

The above algebraic considerations yield the following geometric result (whose proof is parallel to the construction of normalizations and hence the details of which I leave to you): if X is an irreducible abstract algebraic set and K' is a finite extension of $k(X)$, then there exists an irreducible *normal* abstract algebraic set Z equipped with a *finite* surjective map $Z \to X$ and an isomorphism $\iota_Z : K' = k(Z)$ over $k(X)$ such that for any *normal* irreducible abstract algebraic set Y, any dominant map $Y \to X$, and any injection $k(Z) = K' \hookrightarrow k(Y)$ over $k(X)$, there exists a unique dominant map $Y \to Z$ over X inducing the given function field map $k(Z) \hookrightarrow k(Y)$ over $k(X)$. In concrete terms, over open affines $\mathrm{MaxSpec}(A)$ in X one forms the integral closure of A in the finite extension K' of the fraction field $k(X)$ of A and then one uses mapping properties to glue these into a global object with the required properties. We call Z the *normalization of X in the finite extension K' of* $k(X)$, and from transcendence degree considerations with its function field, we see that the irreducible Z has the same dimension as X! You should visualize Z as obtained from gluing integral closures in K' of the coordinate rings of the non-empty open affines in X (whose fraction fields are all naturally identified with $k(X)$).

I again emphasize that the details of the construction of this more general notion of normalization in a finite extension K' of $k(X)$ are essentially the same as in the earlier case of ordinary normalizations (corresponding to the case $K' = k(X)$), so it is safe to leave these details to you; recall that this all requires the good theory of finite maps (preimages of any open affine is again affine, etc). But I observe a basic property that either comes out of the construction, or can be proven by pure thought via universal mapping properties: if U is a non-empty open in X, then the preimage of U in Z *is* the normalization of U in the initially chosen finite extension K' of $k(U) = k(X)$ which was used to define Z as a normalization of X. This property of normalizations being of local nature on the base space X is important below.

We now apply this generalized normalization construction to the dominant map $C \to \mathbf{P}^1$ constructed above. Let $\overline{C}$ be the normalization of $\mathbf{P}^1$ in the finite extension $k(C)$ of its function field $k(\mathbf{P}^1) = k(t)$. By construction, $\overline{C}$ is irreducible, normal, 1-dimensional, and quasi-compact, with function field $k(C)$. Moreover, it is *finite* over $\mathbf{P}^1$ (inducing the *chosen* function field extension $k(t) \hookrightarrow k(C)$ corresponding to the composite dominant map

$$C \to \mathbf{A}^1 \hookrightarrow \mathbf{P}^1),$$

hence $\overline{C}$ is separated (by the lemma above). Being normal and irreducible of dimension 1, $\overline{C}$ is a *smooth curve.* From the mapping properties of normalizations, we can uniquely fill in the top row of the following commutative diagram, with the top row a dominant map:

$$\begin{array}{ccc} C & \to & \overline{C} \\ \downarrow & & \downarrow \\ \mathbf{A}^1 & \hookrightarrow & \mathbf{P}^1 \end{array}$$

What we need to prove is that the top row is an open immersion and that $\overline{C}$ admits a structure of projective variety (i.e., a closed immersion into a projective space). Let $\pi : \overline{C} \to \mathbf{P}^1$ be the map in the right column, so by the local nature of the normalization construction, $\pi^{-1}(\mathbf{A}^1) \to \mathbf{A}^1$ is the normalization of $\mathbf{A}^1$ in the function field extension $k(\overline{C}) = k(C)$ of $k(t)$. But since C is normal, it is obvious that the *finite* dominant map $C \to \mathbf{A}^1$ identifies C with the normalization of $\mathbf{A}^1$ in the function field extension $k(C)$ of $k(t) = k(\mathbf{A}^1)$. That is, the map $C \to \pi^{-1}(\mathbf{A}^1)$ over $\mathbf{A}^1$ is a map between normalizations of $\mathbf{A}^1$ in a common function field, compatibly with that function field identification. From this it follows by the properties of normalization that *this map* $C \to \pi^{-1}(\mathbf{A}^1)$ over $\mathbf{A}^1$ must be an isomorphism! Thus, the top row $C \to \overline{C}$ in the above commutative diagram is indeed an open immersion (more precisely, it induces an isomorphism of C onto $\pi^{-1}(\mathbf{A}^1)$).

We are now left with the problem of proving that $\overline{C}$ admits a structure of projective variety. Renaming $\overline{C}$ as C (which should cause no confusion), what we must prove is that if C is an irreducible smooth curve and $\pi : C \to \mathbf{P}^1$ is a *finite* map (so in particular, C is separated), then there exists a closed immersion $C \hookrightarrow \mathbf{P}^N$ for some large N. Let $U = \pi^{-1}(\mathbf{P}^1 - \{[1,0]\})$, $V = \pi^{-1}(\mathbf{P}^1 - \{[0,1]\})$ be the parts of C lying over the two basic open affines in $\mathbf{P}^1$. In particular, since π is finite we know from the general theory of finite maps that U and V are *affine*! Choose closed immersions $i : U \hookrightarrow \mathbf{A}^n$, $j : V \hookrightarrow \mathbf{A}^m$, so we have a composite map

$$(\pi, i, j) : U \cap V \to \mathbf{P}^1 \times \mathbf{A}^n \times \mathbf{A}^m \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m$$

(the second of these is an open immersion).

Using two Segre embeddings, there is a closed immersion $\mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N$ for some large N. Since C is an irreducible smooth curve in which $U \cap V$ is a dense open, the composite map

$$U \cap V \to C \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$$

to a projective space extends uniquely to a morphism $C \to \mathbf{P}^N$. But the *dense* open $U \cap V$ lands inside of the *closed* subvariety $\mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m$, so by continuity the image of C in $\mathbf{P}^N$ lies inside of this closed subvariety. That is, we have a unique factorization

$$C \stackrel{\iota}{\to} \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N.$$

It suffices to check that this first map ι is a closed immersion. Composing with projections gives a map

$$C \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n \times \mathbf{P}^m$$

whose graph is a *closed immersion*

$$C \to C \times \mathbf{P}^n \times \mathbf{P}^m$$

(as is the graph of any map to a separated abstract algebraic set). But composing this graph morphism with the *finite* map

$$\pi \times 1 \times 1 : C \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m$$

(whose finiteness follows from that of π, by an old homework), we recover exactly the above map ι, as this can be checked on the dense open $U \cap V$ (why?). Thus, ι is a composite of a closed immersion and a finite map. Hence, ι is at least a finite map (being a composite of two finite maps).

But the composite maps

$$U \hookrightarrow C \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n, \;\; V \hookrightarrow C \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^m$$

coincide with the respective injective (!) composites (of closed immersions followed by open immersions)

$$U \hookrightarrow \mathbf{A}^m \hookrightarrow \mathbf{P}^m, \;\; V \hookrightarrow \mathbf{A}^n \hookrightarrow \mathbf{P}^n,$$

as this can be checked on the dense open $U \cap V$ in each of U and V. We conclude that the *finite* map

$$\iota : C \to \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m$$

is injective. Indeed, if $x, x' \in C$ have the same image in the triple product, then by composing with the first projection we conclude that at least $\pi(x) = \pi(x')$ in $\mathbf{P}^1$, so either $x, x' \in U$ or $x, x' \in V$ (look back at the definitions of U and V). But we just saw that the composites from U to $\mathbf{P}^n$ and from V to $\mathbf{P}^m$ are injective, so indeed $x = x'$. Now that we know ι is a finite injective map, we can conclude it is a closed immersion by the general lemma below, applied to the map ι, the open covering $\{U, V\}$ of C, and the projections from $\mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^m$ to $\mathbf{P}^n$ and $\mathbf{P}^m$.

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Lemma 0.3. *Let $f : X \to Y$ be an injective finite map between abstract algebraic sets and let $\{U_i\}$ be an open covering of X. Suppose that there are maps $g_i : Y \to Z_i$ such that the composite maps $U_i \hookrightarrow X \to Y \to Z_i$ can each be factored as a composite of some finite number of open and/or closed immersions in any order. Then f is a closed immersion.*

Note that a finite injective map, while necessarily a closed embedding on topological spaces (as finite maps are closed), need *not* be a closed immersion. Just consider the finite map $\mathbf{A}^n \to \mathbf{A}^n$ defined by raising coordinates to the pth power, where k has positive characteristic p, or in any characteristic consider the bijective finite map $t \mapsto (t^2, t^3)$ from $\mathbf{A}^1$ to the singular curve $y^2 = x^3$ in $\mathbf{A}^2$.

Proof. Since finite maps send closed sets to closed sets, the hypotheses imply that f is a closed embedding on topological spaces. Thus, we just have to show that for each $x \in X$, any section of $\mathscr{O}_X$ near x lifts (on a possibly smaller neighborhood of x) to a section of $\mathscr{O}_Y$ near $f(x)$ (since f is already a closed embedding topologically, this latter condition is exactly the "local surjectivity" condition for $\mathscr{O}_Y \to f_*\mathscr{O}_X$ in the *abstract* definition of a closed immersion, which we *proved* to be the same as a concrete notion over open affines of the target). It is enough to check that if $x \in U_i$, then any such section lifts (near x) to a section of $\mathscr{O}_Z$ near $(g_i \circ f)(x)$ (as such an element pulls back to a section of $\mathscr{O}_Y$ near $f(x)$ of the desired type). But upon factorizing $g_i \circ f|_{U_i}$ into a composite of open and/or closed immersions in some order, we are reduced to the analogous surjectivity claims for open immersions and closed immersions, both of which are obvious!

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Note that the arguments above show that if $\pi : C \to \mathbf{P}^1$ is a *finite* map and C is irreducible and smooth of dimension 1, then C admits a closed immersion into some $\mathbf{P}^N$ (i.e., C is a projective curve). It is actually true in complete generality that if V is a projective algebraic set (i.e., V is an abstract algebraic set admitting a closed immersion into some $\mathbf{P}^n$, such as a projective variety) and $W \to V$ is a finite map from some abstract algebraic set W, then W is also a projective algebraic set. However, the proof of this requires far more sophisticated techniques than those we have available to us in this course (note that if we replace "projective" with "affine", the assertion being made is an immediate consequence of the good theory of affine/finite maps set up on the homework).

We conclude with an observation about irreducible projective curves $C \hookrightarrow \mathbf{P}^N$ which might have singularities. The normalization construction gives us a finite map $\widetilde{C} \to C$ which is an isomorphism over a dense open, with $\widetilde{C}$ an irreducible 1-dimensional smooth variety. I claim that projectivity of C *forces* $\widetilde{C}$ to already be projective. That is, the above general theory ensures that this rather abstract $\widetilde{C}$ can be realized as a dense open in an irreducible smooth projective curve, but I claim that $\widetilde{C}$ already *is* this curve! For example, if C is a highly singular irreducible curve in $\mathbf{P}^2$, its normalization already is the unique smooth projective curve birational to C.

To prove the projectivity of the normalization $\widetilde{C}$ of an irreducible projective curve C, we let $\widetilde{C} \hookrightarrow C'$ be the open immersion into an irreducible smooth projective curve and we want to prove this is an isomorphism. We may view $\widetilde{C} \to C \hookrightarrow \mathbf{P}^N$ as a rational map from C' to $\mathbf{P}^N$, so it uniquely extends to a morphism $C' \to \mathbf{P}^N$. Since the dense open $\widetilde{C}$ lands inside of the closed subvariety $C \hookrightarrow \mathbf{P}^N$, we get a unique factorization

$$C' \to C \hookrightarrow \mathbf{P}^N.$$

In particular, the open immersion $\widetilde{C} \to C'$ respects the map from each side down to C. Thus, $C' \to C$ is a dominant (even surjective birational) map and hence by normality of C' uniquely factors through the normalization $\widetilde{C} \to C$ of C. This gives us a map $C' \to \widetilde{C}$ over C, and it must be an inverse to the above open immersion in the other direction since the composites in either order give rise to self-maps from $\widetilde{C}$ to itself over C and from C' to itself over C, both of which must be the identity (by function field considerations, since the structure maps $\widetilde{C} \to C$ and $C' \to C$ are birational isomorphisms).

One lacuna in this discussion is a *constructive* algorithm for actually computing the normalization of an irreducible curve (since explicit direct calculations of integral closures is essentially impossible in practice). That is, this entire analysis so far has been totally abstract (even though we were able to prove concrete forms of the existence results in the end). We will make the resolution theorem for curves computationally effective by means of the important geometric construction called a *blow-up*; this construction has played an important role throughout the history of work on the problem of resolution of singularities, and it is a basic tool in algebraic geometry.