Math 120 practice final Solution

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Problem 1

(a) True. Because $(g_1, g_2)(g_1', g_2') = (g_1g_1', g_2g_2') = (g_1'g_1, g_2'g_2) = (g_1', g_2')(g_1, g_2)$

(b) False. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not cyclic.

(c) True. It's by Cauchy's Theorem (P.96).

(c) False, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}$ does not contain an element of order 4.

(d) True. If there exists one element has order 4 then $G \cong \mathbb{Z}_4$. If all elements except the identity have order 2, then pick a, b two distinct elements in G, then ab and ba should be the fourth element in G other than $a, b, identity$. So the group is commutative.

(e) False. S_3 is not commutative.

(f) False. Let $H =< (12) >$ and $G = S_3$. First you can check H is not normal in G. And prove directly $N_G(H) = H$. So $N_G(H)$ is not normal in this example.

(g) True. There is only one such group : $\mathbb{Z}_{17}$

(h) False. The number of 5-sylow subgroups should be congruent to 1 mod 5.

Problem 2

(a) First we want find the image of 1. Try to solve k to make $7 \cdot k = 1 \pmod{50}$ will give you k=43. So $\phi(1) = \phi(7 \cdot 43) = \phi(7) \cdot 43 = 6 \cdot 43 = 3 \pmod{15}$. Then we know for every n in $\mathbb{Z}_{50}$, $\phi(n) = \phi(1 \cdot n) = n \cdot \phi(1) = 3n \pmod{15}$.

(b)
$$Ker(\phi) = \{n \in \mathbb{Z}_{50} \,|\, 3n = 0 \pmod{15}\ \} = 5 \cdot \mathbb{Z}_{50}$$

(c) Image=$3 \cdot \mathbb{Z}_{15}$

Problem 4

Let $G/Z(G) =< x >$ because $G/Z(G)$ is cyclic. Take $\tilde{x}$ in $\phi^{-1}(x)$. We can write every element in G as $\tilde{x}h$, $h \in Z(G)$. Randomly pick two elements in G, we can write them as $\tilde{x}^i h_1$ and $\tilde{x}jh_2$ where $h_1, h_2 \in Z(G)$.

$$\tilde{x}^i \underbrace{h_1\tilde{x}^j}_{h_1\in Z(G)} h_2 = \underbrace{\tilde{x}^i\tilde{x}^j}_{\tilde{x}^i\tilde{x}^j commutes} \underbrace{h_1h_2}_{h_1\in Z(G)} = \tilde{x}^j \underbrace{\tilde{x}^i h_2}_{h_2\in Z(G)} h_1 = \tilde{x}^j h_2 \tilde{x}^i h_1$$

This implies any two elements in G commutes. So G is a commutative group.

Problem 5

No. Take $H =< (123) >$ and $G = S_3$. H is cyclic and normal in S_3. S_3/H has order 2 so it's isomorphic to $\mathbb{Z}_2$ which is commutative. But S_3 is not commutative.

Problem 6

If we write $D_8 = \{a, b | a^4 = b^2 = 1, bab^{-1} = a^{-1}\}$ Define a homomorphism $\phi : D_8 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by sending $a^i b^j$ to (i, j). You can check this is an surjective homomorphism and hte kernel is $Z(D_8) = a^2, id$. By the first isomorphism theorem, $D_8/Z(D_8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Problem 7

Because $45 = 3^2 \cdot 5$, every commutative group of order 45 is isomorphic to $\mathbb{Z}_{45}$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$. 3 is an element $\mathbb{Z}_{45}$ of order 15. (0,1,1) is an element in $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ which has order 15.

Problem 8

$108 = 2^2 \cdot 3^3$. All possible invariant factors are:
$2^2 \cdot 3^3$
$2^2 \cdot 3^2, 3$
$2^2 \cdot 3\ , 3, 3$
$2\ \cdot 3^3\ , 2$
$2\ \cdot 3^2\ , 2 \cdot 3$
$2\ \cdot 3\ , 2 \cdot 3, 3$

Problem 9

Let G =1, 9, 16, 22, 29, 53, 74, 79, 81 be a group under multiplication modulo 91. Because G has 9 elements and G is a subgroup of a commutative group, G must be isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. We just need to know if there is an element of order 9.

By direct computation, $9^2 = 81$, $9^3 \equiv 1 \pmod{91}$. $16^2 = 256 \equiv 74 \pmod{91}$, $16^3 \equiv 1 \pmod{91}$. So 9, 81, 16, 74 have order 3. Because $\mathbb{Z}_9$ has only two element of order 3, G is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Problem 10

Define

$$\begin{aligned} \phi : \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ (x, y) &\to (x(\text{mod } 2), y(\text{mod}2)) \end{aligned}$$

Then ϕ is surjective and $Ker(\phi) = H$. By the first isomorphism theorem, $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Problem 11

G is isomorphic to $\mathbb{Z}_{16}$ or $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

In $\mathbb{Z}_{16}$, there are 2 order 4 elements: 4 and 12. but 4+4=8=12+12 (mod 16). So G is not isomorphic to $\mathbb{Z}_{16}$.

In $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, order 4 elements are: (2,*) and (6, *). * means it can be anything in the group (which is $\mathbb{Z}_2$ is this case). But (2,*)+(2,*)=(4,0)=(6, *)+(6, *). So G is not isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$.

In $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ it's possible because (1,0) and (0,1) have order 4 and (2,0)$\neq$ (0,2).

In $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, order 4 elements are: (1,*, *) and (3, *, *). But all their square = (2,0,0). So G is not isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

There is no order 4 element in $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ So G is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Therefore we can conclude G is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Problem 12

For any $h \in H$ and $k \in K$, $hkh^{-1}k^{-1} \in H \cap K$ because H and K are normal. So $hkh^{-1}k^{-1} = e$, $hk = kh$. So HK is a commutative group.

Problem 13

Let $H =< h \mid h^5 = e >$. For any $g \in G$, let $ord(g) = k$. Because k divides —G—, k has to be an odd number.

Because H is normal, $ghg^{-1} = h^i$ for some $i \in \{0, 1, 2, 3, 4\}$. i may depend on g.

$$\begin{aligned} h =& g^k h g^{-k} = g^{k-1}(ghg^{-1})g^{-k+1} = g^{k-1}h^i g^{-k+1} = g^{k-2}(gh^i g^{-1})g^{-k+2} \\ =& g^{k-2}(ghg^{-1})^i g^{-k+2} = g^{k-2}h^{i^2}g^{-k+2} = \ldots = h^{i^k} \end{aligned}$$

Because h has order 5 and $h = h^{i^k}$, $1 = i^k \pmod 5$. If i=2 or 3 or 4, then k must be even. So i=1. Hence $ghg^{-1} = h$, $gh = hg$. Because we pick arbitrary g, That means h commutes with every $g \in G$. So $H \leq Z(G)$.

Problem 14

By Sylow's theorem, if 7-sylow subgroup is not normal then there are at least 8 7-sylow subgroups. Because —G—¡49, 7-sylow subgoup has order 7. They are cyclic, so every element except ideneity is a generator. Therefore the intesection of two distinct 7-sylow subgroup is only the identity.

So G should contain at least 8*6+1=49 differnent elements, which is a contradiction.

Problem 15

$168 = 8 * \cdot 3 \cdot 7$. If there are k 7-sylow subgroups then k divides 24 and $k \equiv 1 \pmod 7$. So k=8.

Problem 16

$175 = 5^2 \cdot 7$. Using sylow's theorem one can show 5-sylow subgoup H and 7-sylow subgroup K are normal. And HK=G, $H \cap K = e$. By problem 12 G is commutative.

Problem 17

If $|Z(G)| = 4$, $|G/Z(G)| = 15$. One can use sylow's theorem to show every group of order 15 is isomorphic ro $\mathbb{Z}_{15}$, which is cyclic. By problem 4, G is commutative. Then $|Z(G)| = |G| = 60$, contradiction.

Problem 18

By Sylow's theorem every two p-sylow subgroup are conjugate. So the conjugation gives an isomorphism.