

Math 120. Basic Algebra

Final Exam - Practice Questions

Note: These problems are only intended to help you study. Questions based on any material covered in class may appear on the actual test.

1. True or false?
 - (a) If G_1 and G_2 are commutative groups then $G_1 \oplus G_2$ is a commutative group.
 - (b) If G_1 and G_2 are cyclic groups then $G_1 \oplus G_2$ is a cyclic group.
 - (c) Every group of order 100 contains an element of order 5.
 - (c) Every group of order 100 contains an element of order 4.
 - (d) Every group of order 4 is commutative.
 - (e) Every group of order 6 is commutative.
 - (f) If $H \leq G$ then $N_G(H)$ is a normal subgroup of G .
 - (g) Any two groups of order 17 are isomorphic.
 - (h) There exists a group G of order 400 having exactly 8 Sylow 5-subgroups.
2. Suppose that $\varphi: \mathbb{Z}/50\mathbb{Z} \rightarrow \mathbb{Z}/15\mathbb{Z}$ is a group homomorphism such that $\varphi(7) = 6$.
 - (a) Determine the general formula for $\varphi(x)$.
 - (b) What is the kernel of φ ?
 - (c) What is the image of φ ?
3. State the First Theorem of Isomorphisms.
4. Let G be a group. Denote by $Z(G)$ its center. If $G/Z(G)$ is cyclic, show that G is commutative, that is $G = Z(G)$.
5. Let H be a normal subgroup of G . If H and G/H are commutative, must G be commutative?
6. Show that the two groups $D_8/Z(D_8)$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are isomorphic.

7. Prove that any commutative group of order 45 has an element of order 15.
8. How many commutative groups of order 108 are there? List all possible such groups together with their invariant factors.
9. The set $\{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the invariant factors of this group.
10. Let $G = \mathbb{Z} \oplus \mathbb{Z}$. Define:

$$H = \{(x, y) \in G \mid x \text{ and } y \text{ are both even integers}\}.$$
 Show that H is a subgroup of G . Determine the order of G/H . What are the invariant factors of G/H ?
11. Let G be a commutative group of order 16. Suppose that there exists two distinct elements $a, b \in G$ such that $|a| = |b| = 4$ and $a^2 \neq b^2$. What are the invariant factors of G ? Express G as a direct sum of cyclic groups.
12. Let G be a group. Suppose that H and K are two normal subgroups of G such that $H \cap K = \{e\}$. Show that if H and K are commutative then HK must be commutative.
13. Let G be a finite group of odd order. Suppose that H is a normal subgroup of G and $|H| = 5$. Show that $H \leq Z(G)$.
14. Let G be a finite group with $|G| < 49$. Show that if G admits a subgroup H of order 7 then H is a normal subgroup.
15. Suppose that G is a group of order 168. If G has more than one Sylow 7-subgroup, exactly how many does it have?
16. Prove that any finite group of order 175 must be commutative.
17. Let G be a group with 60 elements. Show that the center $Z(G)$ cannot have order 4.
18. Let G be a finite group and p be a prime number such that p divides $|G|$. Show that two distinct Sylow p -subgroups of G must be isomorphic.