MATH 116, SPRING [redacted]
HOMEWORK 2 SOLUTIONS

Pg. 28, 1:

If
$$f(x, y) = u(x, y) + iv(x, y) \tag{1}$$
and
$$g(r, s) = \alpha(r, s) + i\beta(r, s), \tag{2}$$
then
$$g(f(x, y)) = \alpha(u(x, y), v(x, y)) + i\beta(u(x, y), v(x, y)). \tag{3}$$

Using the chain rule, we have

$$(\text{Re } g(f))_x = \alpha_r(u, v)u_x + \alpha_s(u, v)v_x \tag{4}$$
$$(\text{Re } g(f))_y = \alpha_r(u, v)u_y + \alpha_s(u, v)v_y \tag{5}$$
$$(\text{Im } g(f))_x = \beta_r(u, v)u_x + \beta_s(u, v)v_x \tag{6}$$
$$(\text{Im } g(f))_y = \beta_r(u, v)u_y + \beta_s(u, v)v_y. \tag{7}$$

Substituting the Cauchy-Riemann equations for f and g,

$$u_x = v_y \tag{8}$$
$$u_y = -v_x \tag{9}$$
$$\alpha_r = \beta_s \tag{10}$$
$$\alpha_s = -\beta_r, \tag{11}$$

it is straightforward to see that Re $g(f)$ and Im $g(f)$ also satisfy the Cauchy-Riemann equations.

Pg. 28, 2:

Let $f(z) = z^2$. We'd like to show that f satisfies the Cauchy-Riemann equations. Set $z = x + iy$.

$$(x + iy)^2 = x^2 - y^2 + 2xyi. \tag{12}$$

$$\begin{aligned} (\text{Re } f)_x &= 2x && (13)\\ &= (\text{Im } f)_y && (14)\\ (\text{Re } f)_y &= -2y && (15)\\ &= -(\text{Im } f)_x. && (16) \end{aligned}$$

The z^3 case is similar. Just multiply it out and check partials with respect to x and y.

Pg. 28, 4:

Let f be an analytic function such that $|f|^2 = C$. We'd like to show that f is constant. This is obvious if $C = 0$, so let's also assume that $C > 0$. Set $f = u + iv$. Then

$$u^2 + v^2 = C. \tag{17}$$

Differentiating, we obtain

$$\begin{aligned} 2uu_x + 2vv_x &= 0 \qquad (18) \\ 2uu_y + 2vv_y &= 0. \qquad (19) \end{aligned}$$

f is analytic, so we can use the Cauchy-Riemann equations to see that

$$\begin{aligned} 2uu_x - 2vu_y &= 0 \qquad (20) \\ 2uu_y + 2vu_x &= 0 \qquad (21) \\ 2uv_y + 2vv_x &= 0 \qquad (22) \\ -2uv_x + 2vv_y &= 0. \qquad (23) \end{aligned}$$

Setting $w_1 = (2u, -2v)$ and $w_2 = (2v, 2u)$, these four equations give

$$\begin{aligned} \nabla u \cdot w_1 &= 0 \qquad (24) \\ \nabla u \cdot w_2 &= 0 \qquad (25) \\ \nabla v \cdot w_2 &= 0 \qquad (26) \\ \nabla v \cdot w_1 &= 0. \qquad (27) \end{aligned}$$

However, $C > 0$, so at each point z the vectors w_1 and w_2 are nonzero. They are also orthogonal and therefore a basis of $\mathbf{R^2}$. This allows us to deduce from the previous equations that

$$\begin{aligned} \nabla u &= (0, 0) \qquad (28) \\ \nabla v &= (0, 0), \qquad (29) \end{aligned}$$

and we conclude that f is a constant function.